Partial Differentiation and Modelling

January 2024

In this worksheet and the accompanying online session we will explore the concept of partial differentiation and how it is useful in modelling physical processes. This requires a basic understanding of the derivatives of some standard functions and also the chain rule. The first two sets of exercises in this sheet are designed to recap material you have hopefully studied at school. If any of it is unfamiliar then please look at the pages on the BBC Bitesize website (click here) as reference or speak to your teacher. The third set of exercises introduces you to a topic that you will cover later at school, although the method is somewhat ad hoc here. Finally the fourth set of exercises introduces you to partial differentiation - this should hopefully be quite approachable with your current knowledge. Please make sure to have understood what a partial derivative is before attending the online class. We will recap it again but it will help if you put time into understanding it now.

1. **Derivatives** - Find $\frac{df}{dx}$ and $\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$ for each of the following functions
   (a) $f(x) = 4$
   (b) $f(x) = \pi x$
   (c) $f(x) = Ax^m$
   (d) $f(x) = A\cos(x) + B\sin(x)$
   (e) $f(x) = \frac{1}{2}(e^x + e^{-x})$

   where A and m are constants. If you haven't come across e^x it has the property that its derivative is also e^x. By chain rule the derivative of e^{-x} is $-e^{-x}$. The function in part (e) has a special name - it is the 'hyperbolic cosine' function which is often shortened to $\cosh(x)$. There is also a 'hyperbolic sine' function defined as
   $$\sinh(x) = \frac{1}{2}(e^x - e^{-x}).$$
   What do you notice about the first and second derivatives of $\cosh(x)$ and $\sinh(x)$? How does this compare to $\cos(x)$ and $\sin(x)$?

2. **Chain Rule** - Find $\frac{df}{dx} = f'(x)$ and $\frac{d^2f}{dx^2} = f''(x)$ of the following functions. Note that in all of these problems you will need to use the *chain rule*. If you haven't come across it before you can find details on the BBC Bitesize website linked above.
   (a) $f(x) = e^{mx}$
   (b) $f(x) = Ae^{-mx}$
   (c) $f(x) = A\cos(mx)$
   (d) $f(x) = A\sin(\pi x)$
   (e) $f(x) = A\cosh(mx) + B\sinh(mx)$

   where again A and m are constants.

3. **Differential Equations** - The theory of differential equations is deep and you will begin studying it properly later in your A levels. Here we make no attempt to go in to the general theory of how to solve differential equations but instead seek to intuit some key results based on the answers to the questions in the previous exercises.

   A differential equation is, most simply, an equation which features the derivatives of a function. For example,

   $$\frac{df}{dx} = 0,$$

   is a differential equation as it features $f'(x)$. If you have done integration then you will know the solution to this equation is $f = A$ where A is an arbitrary constant. Indeed though we can intuit this from $1(a)$ - it didn't matter that the constant was 4, the derivative of any constant is zero.

   A more complicated differential equation is

   $$xf' = f.$$

   Using $1(c)$ can you find an answer to this problem? (*Hint:* You'll need to choose a value of m!)

   What are the most general solutions to

   $$f'' = -f \quad \text{and} \quad f'' = f?$$

   In both of these you should have *two distinct terms* with different arbitrary constants (A and B). Again look towards question 1 for help!

   Now with reference to the results in question 2, what are the solutions to the following differential equations with their corresponding boundary conditions? (*Hint:* find the most general solution first, then impose the two constraints)

   (a) $f'' = -m^2 f, \quad f(0) = 2, \ f'(0) = 0$

   (b) $f'' = -m^2 f, \quad f(0) = 0, \ f'(0) = 2$

   (c) $f'' = m^2 f, \quad f(0) = 2, \ f'(0) = 0$

   (d) $f'' = m^2 f, \quad f(0) = 0, \ f(1) = 2$

   (e) $f'' = -m^2 f, \quad f(0) = 0, \ f(1) = 0$

   Part (e) is trickier than the others. Imposing $f(0) = 0$ is the same as before but, to avoid just getting $f(x) = 0$ for all x, what restrictions must be put on m?

4. **Partial Derivatives** - Partial differentiation is a skill that is unlikely to be covered at school but if you do a STEM course at university you will almost certainly come across it in your first term. It forms the basis for all mathematical modelling in more than one dimension and as such is a great skill to get to grips with now. It concerns functions of more than one variable. For example, find $f(2,3)$ in the following cases:

   (a) $f(x,y) = 2x + 3y$

   (b) $f(x,y) = x^2 + y^3$

   (c) $f(x,y) = x^2 y$

   (d) $f(x,y) = y^2 x$

   (e) $f(x,y) = \cos(\pi x)\sin(\pi y/2)$

   (f) $f(x,y) = \cosh(xy)$.

   f is now a function of *two independent variables*, x and y. Note since these are independent we can vary x without changing y and vary y without changing x - they do not depend on each other. We could use (x,y) as a coordinate in 2D space as you will see in the online class.

   Now our function f is based on two variables what does it mean to differentiate? What is the meaning of f'? This no longer makes sense because we have multiple variables and $'$ doesn't tell us which one we are differentiating with respect to...

   This is where *partial differentiation* comes in. Let the partial derivative in x be written as

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

   Note the curly 'partial', ∂, rather than the straight, d. This indicates that you should take the derivative with respect to x imagining that the y terms are constant. Similarly f_y indicates that you should take the derivative with respect to y imagining that the x terms are constant. For example, in $4(a)$, $\frac{\partial f}{\partial x} = 2+0$ and in $4(b)$, $f_y = 0 + 3y^2$.

   Find

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial^2 f}{\partial x^2}, \ \frac{\partial^2 f}{\partial y^2}, \ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

   for all of the above functions in this question and again evaluate them at $(x,y) = (2,3)$.

In the online class we'll recap some parts of this problem sheet but I recommend you become comfortable with the general solution to $f'' = m^2 f$ and $f'' = -m^2 f$ as a priority, even if your teacher tells you the answer and you simply check that it works.

During the seminar we will use the techniques covered here to solve the 2D steady state heat equation for $T = T(x,y)$:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

subject to the boundary conditions

$$T(0,y) = 0, \quad T(1,y) = 0, \quad T(x,0) = 0, \quad T(x,1) = sin(\pi x).$$

The solution to this solution will tell us the heat profile of a metal slab given its prescribed heat on the boundaries!