NUMBER THEORY - THE WORLD OF CONTINUED FRACTIONS

FIBONACCI MATHS PROGRAMME - BALLIOL COLLEGE

In this session, we will explore a very beautiful topic with a rich history: *continued fractions*. The goal of this problem sheet is to get familiar with them through lots of example, and observe unexpected patterns and properties that we will investigate in more details in the online session.

Let x be a positive real number. We write $\lfloor x \rfloor$ for the greatest integer less than or equal to x. We will define a simple procedure with far-reaching consequences in number theory.

1. Let $a_1 = \lfloor x \rfloor$. If x is an integer, stop the procedure. Otherwise $x - a_1$ is a number strictly between 0 and 1.
2. Let $x_2 = \frac{1}{x - a_1}$, this is a real number bigger than 1.
3. Let $a_2 = \lfloor x_2 \rfloor$. If x_2 is an integer, stop the procedure. Otherwise $x_2 - a_2$ is a number strictly between 0 and 1.
4. Let $x_3 = \frac{1}{x_2 - a_2}$, this is a real number bigger than 1.
5. Keep going...

Let us give an example. Take $x = \frac{1218}{291}$. Then $a_1 = \lfloor x \rfloor = 4$. We have

$$x - a_1 = \frac{1218}{291} - 4 = \frac{54}{291},$$

so $x_2 = \frac{291}{54}$. Now $a_2 = \lfloor x_2 \rfloor = 5$, so

$$x_2 - a_2 = \frac{291}{54} - 5 = \frac{21}{54}$$

and $x_3 = \frac{54}{21}$. Hence $a_3 = 2$ and $x_3 - a_3 = \frac{12}{21}$, so $x_4 = \frac{21}{12}$. If we continue, we find $a_4 = 1$, $x_5 = \frac{12}{9}$, $a_5 = 1$, $x_6 = 3$ and $a_6 = 3$, at which point the procedure ends.

This allows us to write

$$x = a_1 + \frac{1}{x_2} = a_1 + \frac{1}{a_2 + \frac{1}{x_3}} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{x_4}}} = \cdots = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{x_6}}}}},$$

and since $x_6 = a_6$ we get

$$\frac{1218}{291} = 4 + \frac{1}{5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}}}.$$

The expression on the right is an example of a *simple continued fraction*. Here *simple* means that the numerators of all the fractions are 1 (we will only consider those).

To simplify the notation, instead of $x = 4 + \frac{1}{5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}}}$, it is standard to just write $x = [4, 5, 2, 1, 1, 3]$.

Problem 1. *Compute the continued fraction expansion of each of the following numbers:* $\frac{72}{27}$, $\frac{290}{81}$, $\frac{177}{192}$, $\frac{126}{23}$, $\frac{1024}{15625}$.

Problem 2. *One can show that* $\frac{15475}{2309} = [6, 1, 2, 2, 1, 4, 4, 1, 2, 3]$. *Compute the numbers* $c_1 = 6$, $c_2 = 6 + \frac{1}{1} = [6, 1]$, $c_3 = 6 + \frac{1}{1+\frac{1}{2}} = [6, 1, 2]$, *and so on up to* c_{10}. *Plot these numbers* $c_1, \ldots, c_{10}$ *on a numberline (not necessarily to scale). What do you observe? Compute* $c_i - c_{i-1}$ *for* $2 \leq i \leq 10$. *Do you see any pattern?*

Problem 3. *Consider the following sequence:* $c_1 = 1$, $c_2 = 1 + \frac{1}{1}$, $c_3 = 1 + \frac{1}{1+\frac{1}{1}}$, $c_4 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$, *etc. For* $1 \leq i \leq 10$, *write* c_i *as a single fraction. What do you notice?*

Problem 4. *Can you make sense of the infinite expression*

$$1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}} \quad ?$$

Problem 5. *Using the procedure described in the beginning, find a n i nfinite co ntinued fraction expansion for* $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$ *and* $\sqrt{37}$.

Problem 6 (Optional)**.** *Using a calculator, find t he b eginning o f t he c ontinued f raction expansion of* $e - 1$, *where*

$$e = 2.718281828459\ldots$$

is Euler's number. Compute at least 12 *terms of the expansion* $e - 1 = [1, ?, ?, ?, \ldots]$. *Can you conjecture how this continues?*

Mathematical Institute, Oxford