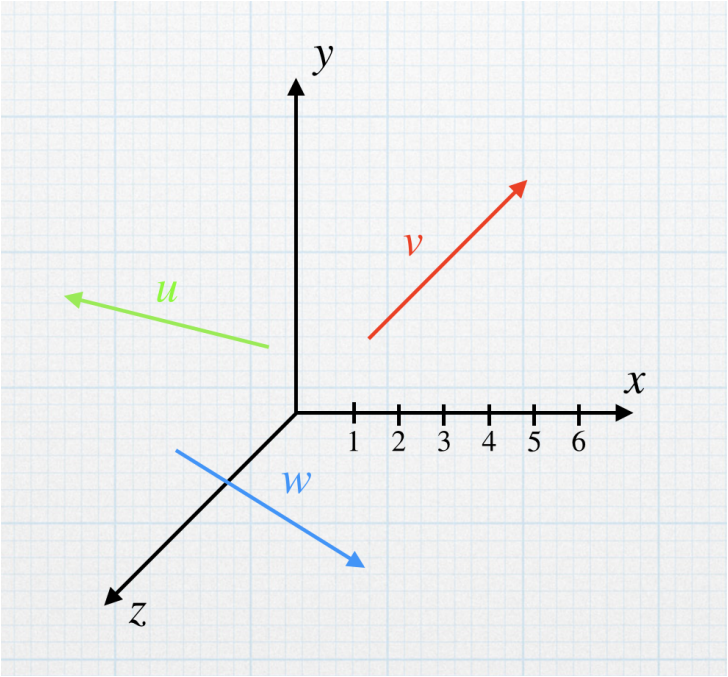
BALLIOL COLLEGE
UNIVERSITY OF OXFORD

# Balliol College Fibonacci Programme 2024




## LINEAR ALGEBRA

Preliminary Exercises

# 1 Basics of vectors and systems of equations

**Exercise 1.1** Let $A$ and $B$ be two points in the real plane $\mathbb{R}^2$ and let $M$ be the midpoint of the segment separating the points $A$ and $B$. Prove that the vectors $\overrightarrow{AM}$ and $\overrightarrow{AB}$ satisfy

$$\overrightarrow{AM} = \frac{1}{2} \cdot \overrightarrow{AB}.$$

**Exercise 1.2** Let $a \in \mathbb{R}$. Solve the system of equations

$$\begin{cases} x - 2y = 1 \\ x + y = a \end{cases}$$

for $x$ and $y$ as a function of the parameter $a$.

# 2 Vector spaces

**Exercise 2.1** Let $E = \mathbb{R}^2$, and consider the subset

$$F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in E \mid x + y = 0 \right\} \subseteq E$$

.

Prove that $F$ is a vector subspace of $E$. *Hint: You need to prove that* $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in F$ *and that if* $v, w \in F$ *and* $\lambda, \mu \in \mathbb{R}$, *then* $\lambda v + \mu w \in F$. *To prove the second point, take two vectors* $v, w \in F$. *Then, using the definition of* $F$, *you know the equation satisfied by the* $x$- *and* $y$-*components of the vectors* $v$ *and* $w$. *Now take* $\lambda, \mu \in \mathbb{R}$ *and verify that the* $x$- *and* $y$-*components of the vector* $\lambda v + \mu w$ *satisfy the same equation as the* $x$- *and* $y$-*components of* $v$ *and* $w$.

**Exercise 2.2** Let $E = \mathbb{R}^2$ and consider the three vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

.

Prove the following:

- $(v_1, v_2)$ is a linearly independent family;
- $(v_1, v_3)$ is a linearly independent family;
- $(v_1, v_2, v_3)$ is *not* a linearly independent family.

Further prove that $(v_1, v_2)$ forms a basis of $E = \mathbb{R}^2$.

*Hint: To prove that* $(v_1, v_2)$ *is a linearly independent family, take two scalars* $c_1, c_2 \in \mathbb{R}$ *and consider the equation* $c_1 v_1 + c_2 v_2 = 0$. *By explicitly writing out the vectors* $v_1$ *and* $v_2$ *using their definition, you obtain a system of equations which you can solve for* $c_1$ *and* $c_2$. *You should find that* $c_1 = c_2 = 0$. *Proceed similarly to prove that* $(v_1, v_3)$ *is a linearly*

*independent family. To prove that $(v_1, v_2, v_3)$ is not a linearly independent family, solve the system of equations associated to the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$ for $c_1, c_2, c_3 \in \mathbb{R}$. You should find that $c_1 = c_2 = c_3 = 0$ is not the only solution to the system of equations. To prove that $(v_1, v_2)$ is a basis, you need to prove that it is both a spanning and linearly independent family, and you should already have proved the second part. For the first part, take an arbitrary vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ and show that you can express it in terms of $v_1$ and $v_2$.*

# 3 Linear maps

**Exercise 3.1** For the following two vector spaces $E$ and maps $f : E \to E$, verify whether $f$ is a linear map or not. If $f$ is a linear map, compute its kernel $\mathrm{Ker}(f)$ and its image $\mathrm{Im}(f)$.

- $E = \mathbb{R}^3$ and $f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x + 3y + z \\ y - z \\ 8x - 6y + z \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E$
- $E = \mathbb{R}[X]$ and $f(Q(X)) = Q(X^2)$ for all $Q \in E$.

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# 4 Matrices

**Exercise 4.1** Let $E$ be the vector space of real $3 \times 2$ matrices and consider

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \\ 8 & -5 \end{pmatrix} \in E\,.$$

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The transpose of a matrix $M$ of size $n \times p$, where $n, p \geq 1$, denoted $M^\intercal$, is the matrix obtained by exchanging the rows and columns of the matrix:

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,p} \\ a_{2,1} & a_{2,2} & \dots & a_{2,p} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,p} \end{pmatrix} \quad \to \quad M^\intercal = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & & \vdots \\ a_{1,p} & a_{2,p} & \dots & a_{n,p} \end{pmatrix}.$$

Write the transpose of the matrix $A$.

**Exercise 4.2** Let $E$ be the vector space of $2 \times 2$ real matrices and let $A, B \in E$ be the matrices given by

$$A = \begin{pmatrix} 6 & 5 \\ 9 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -5 \\ 1 & 12 \end{pmatrix}.$$

Prove the following relations:

- $(A + B)^\intercal = A^\intercal + B^\intercal$;
- $(\lambda A)^\intercal = \lambda A^\intercal$ for all $\lambda \in \mathbb{R}$;
- $(A^\intercal)^\intercal = A$.

*Hint: Compute the left-hand side and the right-hand side of the equations separately. You should find that they are equal.*

**Exercise 4.3** Let $E$ be the vector space of real $2 \times 2$ matrices and consider

$$A = \begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix} \in E\,.$$

.

Prove that $A$ is invertible and find its inverse $A^{-1}$.

**Exercise 4.4** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $E$ be a three-dimensional vector space with basis $(v_1, v_2, v_3)$ and let F be a four-dimensional vector space with basis $(w_1, w_2, w_3, w_4)$. Consider the linear map $f \in \mathcal{L}_{\mathbb{F}}(E, F)$ given by:

$$\begin{aligned} f(v_1) &= 2w_1 + wf_2 - w_3 + 5w_4 \\ f(v_2) &= 15w_2 - 3w_3 + w_4 \\ f(v_3) &= w_1 - 7w_2 + 6w_3 \end{aligned}$$

Give the matrix associated to the linear map $f$.

# 5 Solving systems of linear equations

**Exercise 5.1** Using Cramer's rule, solve the two-dimensional linear system

$$\begin{cases} x + y = 3 \\ x - y = 11\,. \end{cases}$$

**Exercise 5.2** Let $E$ be the vector space of real $2 \times 2$ matrices and consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E$$

where $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc \neq 0$, ie: $A$ is invertible. Use Cramer's rule to compute the inverse of $A$. *Hint: Remember the definition of the invertibility of a matrix in the notes: $A$ is invertible if and only if the equation $AX = Y$ admits a unique solution for any $Y$. If $A$ is a square matrix, the unique solution is given by $X = A^{-1}Y$, where $A^{-1}$ is the inverse matrix of $A$. To solve this question, let $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be any vector in $E$. Solve the equation $AX = Y$ for $X$ and express the solution as $X = BY$ for some matrix $B$. By the definition of invertibility, $B = A^{-1}$ is the inverse of $A$. Check your result by computing the product $AA^{-1}$. This should give you the $2 \times 2$ identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.*