COMBINATORICS - THE SUM-PRODUCT PHENOMENON AND ADDITIVE PATTERNS

FIBONACCI MATHS PROGRAMME - BALLIOL COLLEGE

The focus of this session is to enter the world of *additive combinatorics*, a recent mathematical field that is the subject of much exciting ongoing research.

If X is a finite set, we will write $|X|$ for the number of elements in X.

1. Combinatorics and discrete geometry

Graphs are abstract combinatorial objects. A graph is a collection of *vertices* and *edges*, where each edge connects two distinct vertices. We require that any two vertices are never connected by more than one edge. Graphs can be displayed in the plane: a *drawing* of a graph is a representation of it where its vertices are drawn as points in the plane, and its edges are continuous curves joining the corresponding vertices. Curves are not allowed to have self-intersections, but two distinct curves may intersect. The *number of crossings* of a graph drawing is the number of such intersections. Note that one graph may have many different drawings with different numbers of crossings.

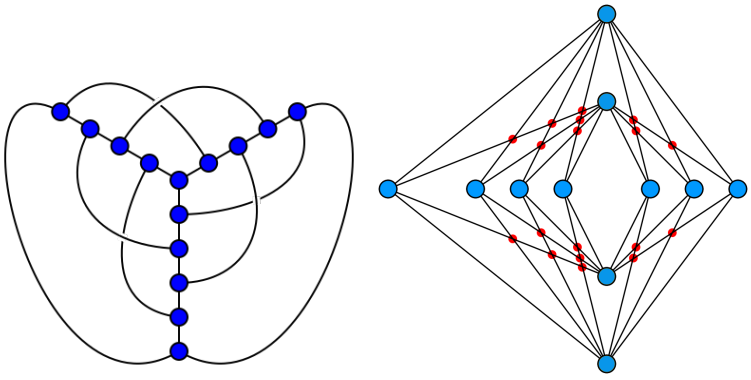


Figure 1. Left: a drawing of a graph with 3 crossings. Right: a drawing of another graph with 18 crossings (vertices in blue, crossings in red).

A graph is *planar* if it admits a drawing with no crossing. We call such a drawing a *plane* drawing.

Problem 2. *Let G be a planar graph with $n \geq 3$ vertices. Consider a plane drawing of G and suppose that it is impossible to add an edge to this drawing without creating a crossing. In other words, all 'faces' (including the 'outer face') are triangles*[1]*.*

Prove that the number of triangles is $e - n + 2$, where e is the number of edges of G.

Hint: show that this is true for $n = 3$ and use induction.

[1] In this context, a *triangle* is a region of the plane whose boundary consists of three edges.

Problem 3. *Let G be a planar graph with $n \geq 3$ vertices. Prove that G has at most $3n - 6$ edges.*

Hint: use the previous problem and count the number of triangles in a different way.

Let L be a finite set of lines and P a finite set of points in the plane. We define the *number of incidences* $I(L, P)$ between L and P to be the number of pairs (ℓ, p) where ℓ is a line in L and p a point of P that lies on ℓ. In other words,

$$I(L, P) = \sum_{\ell \in L} \sum_{p \in P} \mathbf{1}_{p \in \ell}$$

where $\mathbf{1}_{p \in \ell}$ means 1 if p lies on ℓ, and 0 otherwise.

Problem 4 (Optional)**.** *Let L be a finite set of lines and P a finite set of points in the plane. Prove that*

$$I(L, P)^2 \leq |L||P|^2 + |L| \cdot I(L, P).$$

Conclude that

$$I(L, P) \leq \max(2|L|^{1/2}|P|,\ 2|L|).$$

Hint: you may find the Cauchy-Schwarz inequality[2] *useful.*

5. Sets avoiding additive patterns

Write $[N]$ for the set of integers $\{1, 2, \ldots, N\}$. Three integers a, b, c are in *arithmetic progression* (in this order) if $b = \frac{a+c}{2}$.

Problem 6. *Consider the following sequence:*

$$0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, \ldots$$

The rule to construct it is as follows.

- *Start with the first term,* 0.
- *Then, add the smallest possible positive integer not appearing before and which does not create an arithmetic progression with two previous terms of the sequence.*

For example, 1 *is the second term of the sequence, but* 2 *cannot be chosen because* $0, 1, 2$ *is an arithmetic progression.*

Do you see any patterns in this sequence? What number comes after 40*?*

Hint: write the elements of this sequence in a different base.

The sequence described in the previous problem is called the *greedy sequence* avoiding three-term arithmetic progressions. In the next few problems, we will give another construction of sets avoiding the additive pattern $a - 2b + c = 0$.

Problem 7. *Let $N \geq 2$ be an integer. For $1 \leq r \leq 3N^2$, let S_r be the set of all vectors $(a_1, a_2, a_3) \in [N]^3$ (meaning that each coordinate a_i is in $[N]$) such that*

$$a_1{}^2 + a_2{}^2 + a_3{}^2 = r.$$

[2]The Cauchy-Schwarz inequality states that, for any finite set A and any functions f and g taking values in the set of real numbers,

$$\sum_{a \in A} f(a)g(a) \leq \left(\sum_{a \in A} f(a)^2 \right)^{1/2} \left(\sum_{a \in A} g(a)^2 \right)^{1/2}.$$

(a) *Let $1 \leq r \leq 3N^2$. Explain why S_r does not contain three distinct vectors in arithmetic progression*[3].

(b) *Prove that there exists some integer r with $1 \leq r \leq 3N^2$, such that S_r contains at least $\frac{1}{3}N$ many points.*

Problem 8. *Let $N \geq 2$ be an integer.*

Define the function $f : [N]^3 \to \mathbb{R}$ by

$$f(a_1, a_2, a_3) = a_1 + (10N)a_2 + (10N)^2 a_3.$$

(a) *Show that $1 \leq f(a_1, a_2, a_3) \leq (10N)^3$ for all $(a_1, a_2, a_3) \in [N]^3$.*

(b) *Show that, if (a_1, a_2, a_3) and (b_1, b_2, b_3) are two distinct vectors in $[N]^3$, we have*

$$f(a_1, a_2, a_3) \neq f(b_1, b_2, b_3).$$

(c) *Let $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \in [N]^3$. Let $n_1 = f(a_1, a_2, a_3)$, $n_2 = f(b_1, b_2, b_3)$ and $n_3 = f(c_1, c_2, c_3)$. Suppose that n_1, n_2, n_3 are integers in arithmetic progression. Prove that a_i, b_i, c_i forms an arithmetic progression for all $i = 1, 2, 3$.*

Problem 9. *Let $N \geq 2$ be an integer. Prove that there is a subset A of $[(10N)^3]$ of size $|A| \geq \frac{1}{3}N$ and such that three distinct elements of A are never in arithmetic progression.*

Hint: use the two previous problems.

Problem 10 (Optional)**.** *Let $N, d \geq 2$ be integers. Prove that there is a subset A of $[(10N)^d]$ of size $|A| \geq \frac{1}{d}N^{d-2}$ and such that three distinct elements of A are never in arithmetic progression.*

Problem 11 (Optional)**.** *Let $\varepsilon > 0$ be a small real number. Prove that there exists an integer $M \geq 1000$ and a subset $A \subset [M]$ such that $|A| \geq M^{1-\varepsilon}$ and such that three distinct elements of A are never in arithmetic progression.*

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[3]Three vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{R}^3$ are in arithmetic progression if $\mathbf{v_1} - 2\mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$, where this is an equality of *vectors* (so the right-hand side here is the zero vector $(0, 0, 0)$).