Hyperbolic functions and differentiation

Module M4.6

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1 Opening items

1.1 Module introduction

Figure 1 shows a situation that you have certainly seen countless times, a heavy cable suspended between two pylons. We all know that the cable sags in the middle, but is there a mathematical function which describes the shape of this curve? It turns out that there is such a function, and it is particularly relevant to this module because it is one of the so-called *hyperbolic functions*. The hyperbolic functions: sinh (x), cosh (x), tanh (x), sech (x), arctanh (x) and so on, which have many important applications in mathematics, physics and engineering, correspond to the familiar trigonometric functions: sin (x), cos (x), tan (x), sec (x), arctan (x), etc.

In Section 2 of this module we begin by defining the basic hyperbolic functions sinh (x), cosh (x) and tanh (x), and show how the infinite series for these functions are related to those of the corresponding trigonometric functions. We also show how these two sets of functions are related through the introduction of the complex number, i (where $i^2 = -1$). Section 2 concludes with a description of how the first few terms of the infinite series for the basic hyperbolic functions can be used to provide approximations to the functions for small values of their arguments. In Section 3 we go on to consider more advanced aspects of hyperbolic functions, including the reciprocal and inverse functions. Section 4 lists some useful identities which are analogous to those listed elsewhere in *FLAP* for the trigonometric functions. We end, in Section 5, by finding derivatives of some of the hyperbolic functions, which also provides practice in using differentiation techniques. The final example given in Section 5 leads to the function that describes the shape of a heavy cable suspended between two pylons (illustrated in Figure 1).

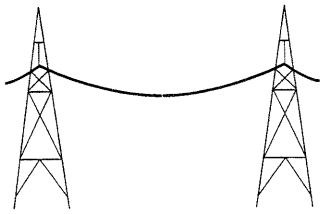


Figure 1 A heavy cable suspended between two pylons.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 7. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 6.1) and the *Achievements* listed in Subsection 6.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 6.3. If you have difficulty with only two of the questions, you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 Define sinh (x), cosh (x) and tanh (x) in terms of the exponential function, $\exp(x) = e^x$.

Question F2 First define sech (x) and coth (x), and then use your answer to the previous question to show that

$$\text{sech}\,(x) = \sec\,(ix) \quad \text{and} \quad \coth\,(x) = i\cot\,(ix)$$

Question F3 Show that

$$\text{arccoth}\,(x) = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) \quad \text{for } |x| > 1$$

Use the fact that $\log_e(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right\}$ for $|x| < 1$

to find a series for arccoth $(1/s)$ valid for $|s| < 1$, and then use this series to estimate the value of arccoth (10).

Use the logarithmic form for arccoth (x) given above, and your calculator, to check your estimate for arccoth (10).

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Question F4 Use the logarithmic expression for arccoth (x) (see Question F3) to find $\frac{d}{dx}\text{arccoth}\,(x)$.

hyperbolic functions
sinh
cosh

1.3 Ready to study?

Study comment To begin the study of this module you need to be familiar with the following: the definition of a *function* and the general formula for the solution of a *quadratic equation*; also the *exponential function*, e^x, including its Taylor *series* expansion about $x = 0$ and the main features of its graph (the function e^x is often written as $\exp(x)$), plus the *trigonometric functions*, including their *graphs*, *series*, *identities*, *reciprocal* and *inverse functions*. You will also need to have some skill at *differentiation* (including the *quotient rule* and *chain rule*), and have some familiarity with *complex numbers* (including *multiplication* and *addition* of such numbers, and the use of *Euler's formula* $e^{i\theta} = \cos\theta + i\sin\theta$). It would also be helpful, though not essential, if you have met *odd* and *even* functions, and *factorials*. If you are unfamiliar with any of these terms, refer to the *Glossary*, which will indicate where they are developed in *FLAP*. The following *Ready to study questions* will help you to establish whether you need to review some of the above topics before embarking on this module.

Throughout this module we use $\sqrt{x}$ to denote the *positive* square root of x; for example, $\sqrt{4} = 2$.

Question R1 (a) Write down the Taylor expansion of e^x about $x = 0$.

(b) Write down the solutions of the quadratic equation $x^2 - ax + 1 = 0$ where $a \geq 2$. Hence write down the solutions of the equation $e^{2w} - ae^w + 1 = 0$.

You will find the answers to these questions in Section 7.

Question R2 Use your calculator to find the values of:

(a) $\arcsin(1/3)$ and $\dfrac{1}{\sin(1/3)}$, (b) $\sin[\sin(1/3)]$ and $\sin^2(1/3)$.

You may often see (although not in *FLAP*) the notation $\sin^{-1}$ used in place of arcsin.

Question R3 If $s(x)$ and $c(x)$ are defined by:

$$s(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad c(x) = \frac{e^{ix} + e^{-ix}}{2}$$

show that: (a) $s(x) \times s(x) + c(x) \times c(x) = 1$.

(b) Use Euler's formula to express $s(x)$ and $c(x)$ in more familiar form.

Question R4 Express $\dfrac{d}{dx}\left(\dfrac{u(x)}{v(x)}\right)$ and $\dfrac{d}{dx}g(f(x))$ in terms of derivatives of u, v, g and f.

2 Basic hyperbolic functions

2.1 Defining sinh, cosh and tanh

It follows from Euler's formula (see Question R3) that the trigonometric functions sine and cosine can be expressed in terms of the exponential function as

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{1}$$

and

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{2}$$

Since analogous combinations of e^x and e^{-x} occur frequently in mathematics and physics, it is useful to define the **hyperbolic functions**, **sinh** (x) and **cosh** (x), by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{3}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{4}$$

sinh and cosh are pronounced as 'shine', 'cosh'; while tanh is pronounced 'than' (with a 'th' as in the word 'thin').

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These functions are given their particular names because they have much in common with the corresponding trigonometric functions, but we leave a discussion of this until later.

◆ Write down (without using your calculator) the values of sinh (0) and cosh (0).

◇ Since $e^0 = 1$ we have sinh (0) = 0 and cosh (0) = 1. □

Question T1 Use the basic definitions of cosh (x) and sinh (x) in terms of exponentials to show that

(a) sinh (x) is an odd function while cosh (x) is an even function

(b) $e^x = \cosh(x) + \sinh(x)$

(c) $\cosh^2(x) - \sinh^2(x) = 1$ □

It is also convenient to define a third hyperbolic function, **tanh** (x), by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad (5)$$

◆ What is the value of tanh (0)?

◇ tanh (0) = 0 since (as we saw before) sinh (0) = 0 and cosh (0) = 1. □

The graphs of these three basic hyperbolic functions are given in Figures 2 to 4. Notice in Figure 3 that cosh $(x) \geq 1$. In Figures 2 and 3 we see that sinh (x) and cosh (x) are very close in value when x is large and positive, but sinh $(x) <$ cosh (x) for other values of x. From Figure 4 we see that tanh (x) is approximately 1 when x is large and positive, and approximately -1 when x is large and negative, and that $-1 <$ tanh $(x) < 1$.

Why are they called *hyperbolic* functions?

To answer this question, we define variables x and y by

$$x = a\cosh(s) \qquad (6)$$

$$y = b\sinh(s) \qquad (7)$$

Note The equations for x and y in terms of s are known as *parametric equations* since s is a parameter which specifies a point on a curve (in this case, a hyperbola).

Then use Question T1(c) to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2(s) - \sinh^2(s) = 1 \qquad (8)$$

which is the equation for a hyperbola, as shown in Figure 5. Thus Equations 6 and 7 are the parametric equations of a hyperbola, and this is the reason why sinh and cosh are known as hyperbolic functions. As we shall see, there are many other functions (such as tanh) which can be defined in terms of sinh and cosh and these are all collectively known as *hyperbolic functions.*

tanh

You will find the answers to all *Text questions* in Section 7.

A function $f(x)$ is

odd if $f(x) = -f(-x)$

even if $f(x) = f(-x)$ for all values of x.

The same notation is used for positive powers of *hyperbolic functions* as for *trigonometric functions*. For example, $\sinh^2(x)$ means $\sinh(x) \times \sinh(x)$.

We will refer back to the result given in Question T1 part (c) shortly.

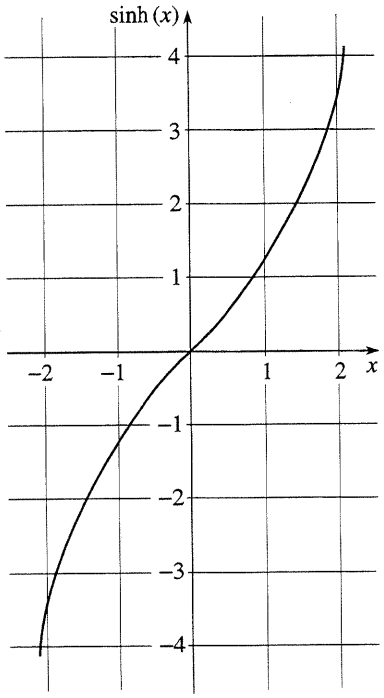


Figure 2 Graph of sinh (x).

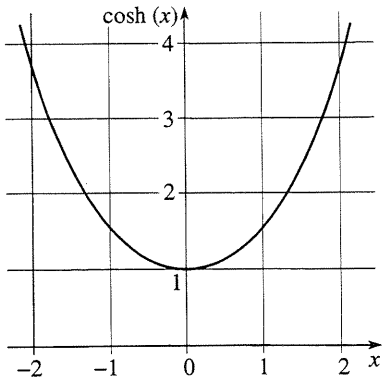


Figure 3 Graph of cosh (x).

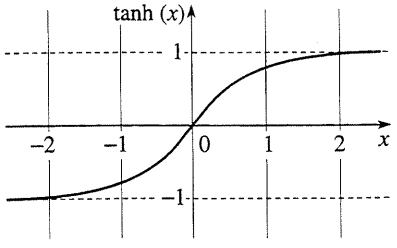


Figure 4 Graph of tanh (x).

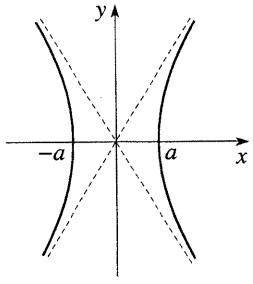


Figure 5 The hyperbola corresponding to $x^2/a^2 - y^2/b^2 = 1$.

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Notice that, unlike the corresponding trigonometric functions, the sinh, cosh and tanh functions are not periodic. Moreover, sinh (x) and cosh (x) may take very large values (unlike sin (x) and cos (x)).

◆ Use the exponential function on your calculator to find sinh (1), cosh (1) and tanh (1). From these results calculate $\cosh^2(1) - \sinh^2(1)$.

◇ $$\sinh(1) = \frac{e - e^{-1}}{2} \approx \frac{2.71828 - 0.36788}{2} \approx 1.175$$

$$\cosh(1) = \frac{e + e^{-1}}{2} \approx \frac{2.71828 + 0.36788}{2} \approx 1.543$$

$$\tanh(1) = \frac{\sinh(1)}{\cosh(1)} \approx \frac{1.175}{1.543} \approx 0.762$$

and $$\cosh^2(1) - \sinh^2(1) \approx (1.543)^2 - (1.175)^2 \approx 1.00022$$ □

Note A more accurate result for this calculation (and also for the calculation of tanh (1)) could be obtained by retaining as many decimal places as possible in the intermediate steps. In fact, from Question T1(c) we know that the result for $\cosh^2(1) - \sinh^2(1)$ should be exactly 1.

Question T2 The speed of waves in shallow water is given by the following equation

$$v^2 = A\lambda \tanh\left(\frac{6.3d}{\lambda}\right)$$

where λ is the wavelength and d is the depth of the water. Find the speed, v when $d = 6$ m, $\lambda = 18.9$ m and $A = 1.8\ \text{m s}^{-2}$. □

2.2 Series for sinh, cosh and tanh

So far we have defined the three basic hyperbolic functions in terms of the exponential function, e^x, which in turn can be defined in terms of an infinite series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (9)$$

The right-hand side of Equation 9 is actually a Taylor expansion about $x = 0$ (as are the other series considered in this subsection). An introduction to the *Taylor expansion* is given elsewhere in *FLAP*.

Replacing x with $-x$ this equation gives us

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \qquad (10)$$

and hence Equation 3 gives

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \qquad (11)$$

Comparing Equation 11 with the series for sin (x)

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \qquad (12)$$

we see that both series contain only odd powers (they are both *odd* functions) and the only difference is that the signs do not alternate for the sinh series. Likewise, from Equation 4, we find that the series for cosh (x) consists of the *even* terms in the series for e^x; that is

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \qquad (13)$$

and comparing this with the series for cos(x)

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \qquad (14)$$

we again see that they both contain even powers (both are *even* functions) and the only difference is that the signs do not alternate for the cosh function.

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1.1

There is a similar relationship between tan (x) and tanh (x) since it can be shown that

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots \quad (15)$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \quad (16)$$

The complete series for tanh (x) and tan (x) involve Bernoulli numbers which are not covered in *FLAP*. Consequently only the first few terms of each series are given here.

The Bernoulli numbers can be defined by the series

$$x/(e^x - 1) = 1 - x/2 + B_1 x^2/2! - B_2 x^4/4! + B_3 x^6/6! + \ldots$$

We can now appreciate the motivation for giving the hyperbolic functions their names; in each case the 'h' is added to give the 'hyperbolic' version of the related trigonometric function.

You are probably familiar with the many trigonometric functions that can be defined in terms of the sine and cosine functions, and, as you might expect, a large number of hyperbolic functions can be similarly defined in terms of sinh and cosh (as we will see in Section 3).

◆ Write down and simplify the Taylor expansions about $x = 0$ for
(a) sinh (x) + sin (x), and (b) cosh (x) − cos (x).

◇ (a) Using Equations 11 and 12 we have

$$\sinh(x) + \sin(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots + \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right]$$

$$= 2\left[\frac{x}{1!} + \frac{x^5}{5!} + \frac{x^9}{9!} + \ldots\right] = 2\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!}$$

(b) Similarly from Equations 13 and 14 we find

$$\cosh(x) - \cos(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\right]$$

$$= 2\left[\frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \ldots\right] = 2\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \quad \square$$

Question T3 Use the series for sinh (x) and cosh (x) to show that

$$\frac{d}{dx}\sinh(x) = \cosh(x) \quad (17)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \quad (18) \ \square$$

2.3 Connection with sine, cosine and tangent via complex numbers

The study of complex numbers (that is, numbers involving i where $i^2 = -1$) leads to the following important result, known as Euler's formula, which connects the exponential, sine and cosine functions

$$e^{ix} = \cos(x) + i\sin(x) \quad (19)$$

By replacing x by $-x$ we have $e^{-ix} = \cos(x) - i\sin(x)$

from which we can obtain concise expressions for cos (x) and sin (x) which resemble the definitions of cosh (x) and sinh (x) given in Subsection 2.1.

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (20)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad (21)$$

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reciprocal hyperbolic functions
cosech
sech
coth

This suggests that identities should exist which relate the hyperbolic and trigonometric functions. Such identities can be obtained if we replace x by ix in the basic definitions of cosh (x) and sinh (x)

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2}$$

and then write these expressions in terms of cos (x) and sin (x).

$$\sinh(ix) = i\sin(x) \tag{22}$$

$$\cosh(ix) = \cos(x) \tag{23}$$

Question T4 Use the expressions for the sine and cosine functions in terms of the exponential function to relate (a) sin (ix) to sinh (x), and (b) cos (ix) to cosh (x). □

The connection between the hyperbolic and trigonometric functions can also be seen from their series. For example

$$\begin{aligned}\sinh(ix) &= \frac{ix}{1!} + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \frac{(ix)^7}{7!} + \frac{(ix)^9}{9!} + \ldots \\ &= \frac{ix}{1!} - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - \ldots \\ &= i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right] = i\sin(x)\end{aligned}$$

Question T5 Use the series for cos (x) and cosh (x) to show that cos (ix) = cosh (x). □

2.4 Small argument approximations

Since the infinite series for sinh (x) and cosh (x) given in this module are Taylor expansions about $x = 0$, we would expect the first few terms in the series to provide reasonable approximations to the functions for values of x 'close' to zero. The following example provides an illustration.

Example 1 Use Equation 11 to evaluate sinh (0.2) to three places of decimals.

Solution From Equation 11 we have

$$\sinh(0.2) = 0.2 + \frac{0.008}{3!} + \frac{0.00032}{5!} + \ldots = 0.2 + 0.0013 + \ldots \approx 0.201$$

This is consistent with my calculator which gives a value of 0.201 336. □

The usual (and somewhat suspect) approach adopted by physicists is to stop calculating when terms in the series no longer contribute to the required accuracy. Accurate approximations for the error do exist, but they are beyond the scope of *FLAP*.

Question T6 Use the series given Equation 15 to find tanh (0.4) to three places of decimals. Check the result with your calculator. □

Your calculator may not have buttons for hyperbolic functions. If it doesn't, then use the basic definition of tanh (x) in Equation 5 together with the e^x or EXP button.

3 More advanced hyperbolic functions

3.1 Reciprocal hyperbolic functions

The **reciprocals** of the basic hyperbolic functions occur fairly frequently and so are given special names **cosech**, **sech** and **coth**:

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \text{provided } x \neq 0 \tag{24}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \text{for all } x \tag{25}$$

$$\coth(x) = \frac{1}{\tanh(x)} \quad \text{provided } x \neq 0 \tag{26}$$

Some computer algebra packages use csch in place of cosech.

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Notice that cosech (x) is the reciprocal of sinh (x) and that sech (x) is the reciprocal of cosh (x). As in the case of trigonometric functions, this terminology may seem rather odd, but it is easily remembered by recalling that each reciprocal pair — (sinh, cosech), (cosh, sech), (tanh, coth) — involves the letters 'co' just once. In other words, there is just one 'co' between each pair. Also notice that the cosech and coth functions are undefined when their partner functions are zero. By contrast, sech (x) is defined for all real x since cosh (x) is never zero, see Figure 3.

The reciprocal functions can be written in terms of e^x as

$$\text{cosech}(x) = \frac{2}{e^x - e^{-x}} \tag{27}$$

$$\text{sech}(x) = \frac{2}{e^x + e^{-x}} \tag{28}$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{29}$$

and graphs of these functions are given in Figures 6 to 8, respectively.

Question T7 Show that $\coth^2(x) - \text{cosech}^2(x) = 1$. □

3.2 *Inverse hyperbolic functions (and logarithmic forms)*

A common problem is to find what argument gives rise to a particular value of some hyperbolic function. For example, if sinh $(x) = 3.6$, what is the value of x? We can get an approximate value of x by looking at Figure 2.

◆ Use Figure 2 to find the (approximate) value of x that corresponds to sinh $(x) = 3.6$?

◇ x is approximately 2.0. □

To answer such questions more precisely we need to know the **inverse hyperbolic functions** which 'undo' the effect of the hyperbolic functions. These new functions are known as **arcsinh**, **arctanh** and **arccosh**, and the first two functions are defined by

$$\text{arcsinh}\,[\sinh(x)] = x \tag{30}$$

$$\text{arctanh}\,[\tanh(x)] = x \tag{31}$$

Graphs of these functions are given in Figures 9 and 10, respectively. Notice that the domain of arctanh (x) is $-1 < x < 1$.

inverse hyperbolic functions
arcsinh
arccosh
arctanh

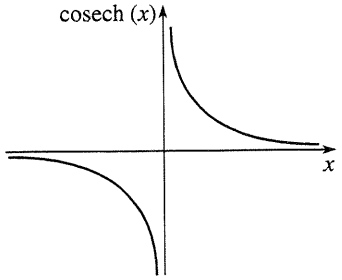


Figure 6 Graph of cosech (x).

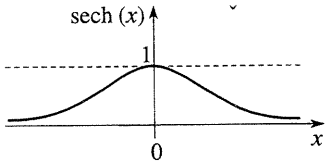


Figure 7 Graph of sech (x).

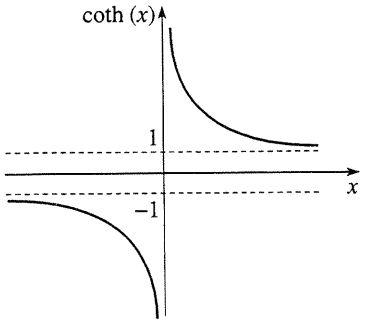


Figure 8 Graph of coth (x).

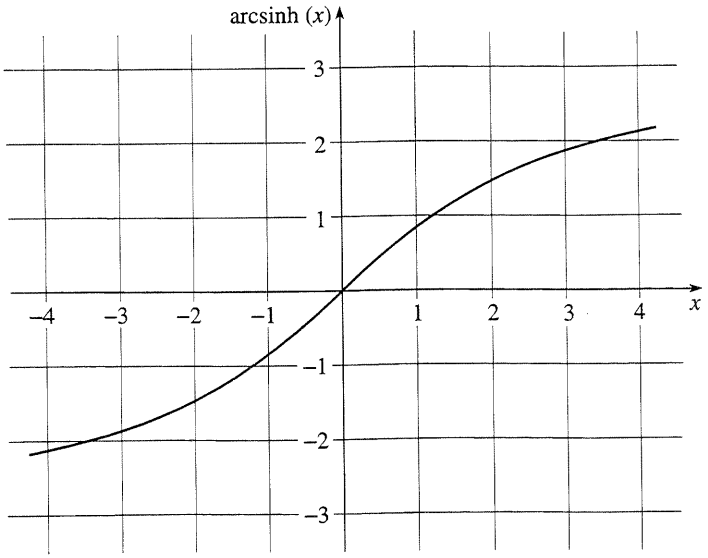


Figure 9 Graph of arcsinh (x).

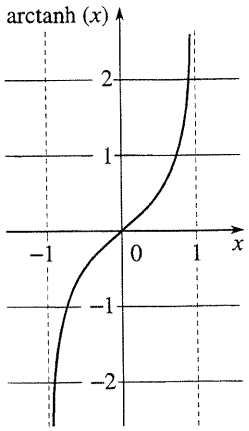


Figure 10 Graph of arctanh (x).

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◆ From Figure 3, estimate values of x that correspond to cosh $(x) = 1$, 2 and 3?

◇ The values of x are approximately 0, ± 1.3 and ± 1.8. □

From Figure 3, we see that for any given value of cosh $(x) > 1$ there are *two* corresponding values of x; in other words, the equation cosh $(x) = a > 1$ has two solutions. Consequently, in order for arccosh to be a function, we must impose a condition which picks out just one value (i.e. $x \geq 0$). Figure 11 shows the graph of the inverse function (the continuous line), and as you can see we choose the positive value for arccosh (x), which gives

This is precisely the same problem that we have when defining the inverse of the function $F(x) = x^2$. We solve it by assigning the unique positive value to the square root in order to make $\sqrt{x}$ a function. Remember that a function must give rise to a unique value $f(x)$ for each value of x.

$$\text{cosh [arccosh } (x)] = x \quad \text{for } x \geq 1 \tag{32a}$$

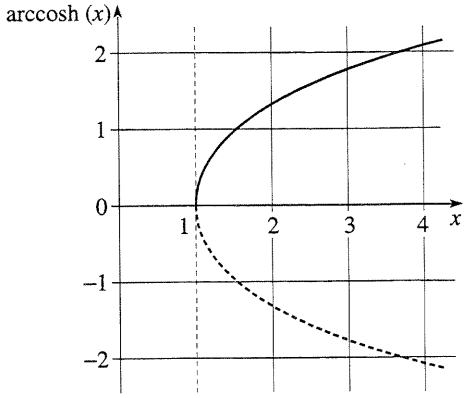


Figure 11 Graph of arccosh (x). Note that only the solid line is involved in the definition.

◆ (a) Simplify $\sqrt{x^2}$ and $\left(\sqrt{x}\right)^2$. Do you need to place any restrictions on the values of x in either case?

(b) Simplify arccosh [cosh (x)] and cosh [arccosh (x)]. Do you need to place any restrictions on the values of x in either case?

◇ (a) In this module we always use the positive square root, so that $\sqrt{x^2} = |x|$ and there are no restrictions on the value of x. On the other hand, $\sqrt{x}$ only makes sense if $x \geq 0$, in which case $\left(\sqrt{x}\right)^2 = x$.

(b) arccosh always gives rise to a positive value, and cosh (x) is defined for all values of x, so that arccosh [cosh (x)] = $|x|$ for all x. On the other hand, arccosh (x) only makes sense if $x \geq 1$, in which case cosh [arccosh (x)] = x. □

If we agree to restrict ourselves to positive values of x, so that we can be sure that $|x| = x$, there is no problem with the inverse of cosh (just as there is no problem with the inverse of 'squaring' if we restrict ourselves to positive numbers) for then we have

$$\text{arccosh [cosh } (x)] = x \quad \text{for } x \geq 0 \tag{32b}$$

◆ From Figure 3, what are the approximate values of arccosh (2) and arccosh (3)?

◇ We find arccosh (2) ≈ 1.3 and arccosh (3) ≈ 1.8. These values are consistent with Figure 11.

◆ Find all solutions of the equation cosh $(x) = 2$.

◇ Our first thought might be that we have just shown that arccosh (2) ≈ 1.3 and therefore *the* solution is approximately 1.3. However, from Figure 3 we know that there are two solutions of the equation cosh $(x) = 2$. These solutions are given by

$$x = \pm \text{ arccosh } (2) \approx \pm 1.3 \quad \square$$

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arccosech
arcsech
arccoth

As in the case of trigonometric functions, an alternative notation is sometimes used to represent the inverse hyperbolic functions:

$\sinh^{-1}(x)$ for arcsinh (x)

$\cosh^{-1}(x)$ for arccosh (x)

$\tanh^{-1}(x)$ for arctanh (x)

Notice that there is *no* connection with the *positive* index notation used to denote powers of the hyperbolic functions, for example, using $\sinh^2(x)$ to represent $[\sinh(x)]^2$. Also notice that, although this notation might make it appear otherwise, there is still a clear distinction between the inverse hyperbolic functions and the reciprocal hyperbolic functions. So

$$\sinh^{-1}[\sinh(x)] = \text{arcsinh}[\sinh(x)] = x$$

but $$\text{cosech}(x) \times \sinh(x) = \frac{1}{\sinh(x)} \times \sinh(x) = 1$$

More inverse hyperbolic functions

There are three more inverse hyperbolic functions which we have not mentioned yet; these are **arccosech** (x), **arcsech** (x) and **arccoth** (x). As you might expect, they are defined so that

$$\text{arccosech}[\text{cosech}(x)] = x \tag{33}$$

$$\text{arcsech}[\text{sech}(x)] = x \tag{34}$$

$$\text{arccoth}[\coth(x)] = x \tag{35}$$

Graphs of these functions are given in Figures 12 to 14, respectively, and the following exercise leads to some useful identities.

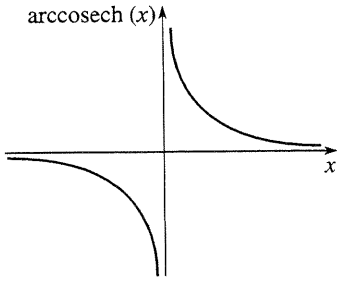


Figure 12 Graph of arccosech (x).

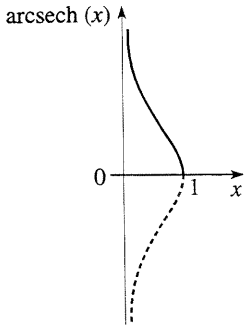


Figure 13 Graph of arcsech (x).

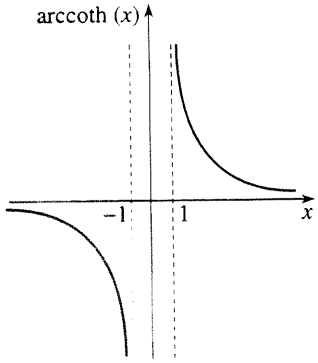


Figure 14 Graph of arccoth (x).

Question T8 Show that (a) arccosech (x) = arcsinh $(1/x)$, (b) arcsech (x) = arccosh $(1/x)$, and (c) arccoth (x) = arctanh $(1/x)$. □

Connection with logarithmic function

Given the relationship between the exponential and logarithmic functions, and the fact that the hyperbolic functions are defined in terms of the exponential function, it might be expected that the inverse hyperbolic functions can be expressed in terms of the logarithmic function. This is indeed the case, and it can be shown that the three basic hyperbolic functions can be written as

$$\text{arcsinh}(x) = \log_e\left(x + \sqrt{x^2 + 1}\right) \tag{36}$$

$$\text{arccosh}(x) = \log_e\left(x + \sqrt{x^2 - 1}\right) \quad \text{for } x \geq 1 \tag{37}$$

$$\text{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \tag{38}$$

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As an example, we now prove the second of these identities.

We start by setting $y = \text{arccosh}\,(x)$

which implies that $x = \cosh(y) = \dfrac{e^y + e^{-y}}{2}$, and we see that $x \geq 1$. (☞)

If $y = \text{arccosh}\,(x)$ then $\cosh(y) = x$.

See Figure 3.

Multiplying the right- and left-hand sides of this last relationship bt $2e^y$ and rearranging, we obtain $e^{2y} - 2x e^y + 1 = 0$,

a quadratic equation in e^y which we can solve to obtain

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

The question now is whether either (or both) of the signs on the right-hand side of this equation are allowable. First we note that the right-hand side must be greater than or equal to one, because $y = \text{arccosh}\,(x) \geq 0$, so $e^y \geq 1$; then we can eliminate the negative sign as follows. (☞)

Alternatively, if we suppose that

$$e^y = x - \sqrt{x^2 - 1},$$

then we would have $x - \sqrt{x^2 - 1} \geq -1$, which would imply $(x-1)^2 \geq x^2 - 1$, which would give $-2x + 1 \geq -1$, or $x \leq 1$. But we know that $x \geq 1$, so this is a contradiction.

We consider the graphs of $Y_1 = \left(x + \sqrt{x^2 - 1}\right)$ and $Y_2 = \left(x - \sqrt{x^2 - 1}\right)$

for $x \geq 1$ (shown in Figure 15), and ask if it is possible for either Y_1 or Y_2 to be greater than or equal to 1 as required.

From the graphs it is clear that $Y_2 \leq 1$, which makes this choice impossible, and therefore the only acceptable solution is

$$e^y = x + \sqrt{x^2 - 1}$$

Taking the logarithm of this result gives us

$$y = \log_e\left(x + \sqrt{x^2 - 1}\right)$$

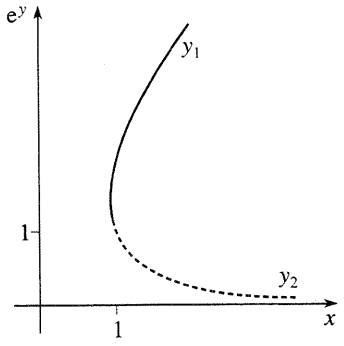


Figure 15 Graphs of $Y_1 = \left(x + \sqrt{x^2 - 1}\right)$ (full line) and $Y_2 = \left(x - \sqrt{x^2 - 1}\right)$ (dotted line) for $x \geq 1$.

The remaining two identities can be proven in a similar way, as the following question demonstrates.

Question T9 Show that

$$\text{arctanh}\,(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \quad \square$$

Expressing the inverse hyperbolic functions in terms of the logarithmic function can be a useful way of solving certain algebraic equations. For example, suppose we want the values of x which satisfy

$$\sinh^2(x) - 3\cosh(x) + 3 = 0$$

then we can use the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

to obtain an equation involving just cosh (x)

$$\cosh^2(x) - 3\cosh(x) + 2 = 0$$

This can be factorized to give

$$(\cosh(x) - 2)(\cosh(x) - 1) = 0$$

so, either $x = \pm\,\text{arccosh}\,(2)$ (☞)

Don't forget that in general there are two solutions to equations such as cosh $(x) = 2$.

or $x = \pm\,\text{arccosh}\,(1)$

Using $\text{arccosh}\,(x) = \log_e\left(x + \sqrt{x^2 - 1}\right)$

gives $x = \pm\,\text{arccosh}\,(2) = \pm\log_e\left(2 + \sqrt{3}\right)$

or $x = \pm\,\text{arccosh}\,(1) = \pm\,\log_e(1) = 0$

(One advantage of this approach is that the logarithmic function is more likely to be available on a calculator.)

Question T10 Solve $\cosh^2(x) + \sinh(x) = 3$, in terms of the logarithmic function. $\square$

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4 Identities

hyperbolic function identities
symmetry relations
basic identities
addition identities
double-argument identities
half-argument identities
sum identities

There are many identities involving trigonometric functions, and there are an equally large number of identities involving hyperbolic functions. In fact, since the trigonometric and hyperbolic functions are related (by, for example, $\sinh(ix) = i\sin(x)$) every trigonometric identity has an analogue for hyperbolic functions. The most important **hyperbolic function identities** are listed below, and it is worth comparing them with their trigonometric partners which are given elsewhere in *FLAP*.

The **symmetry relations:** (☞)

The symmetry relations show the oddness and evenness of the function.

$$\sinh(-x) = -\sinh(x) \tag{39}$$

$$\cosh(-x) = \cosh(x) \tag{40}$$

$$\tanh(-x) = -\tanh(x) \tag{41}$$

It is not suggested that you should be able to quote all of these identities. Most people just learn the few that they use most frequently and look the rest up in tables of such identities. The really important point is to know what kind of identities do exist.

The **basic identities:**

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \tag{42}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{43}$$

$$\coth(x) = \frac{\operatorname{cosech}(x)}{\operatorname{sech}(x)} \quad \text{for } x \neq 0 \tag{44}$$

$$1 - \tanh^2(x) = \operatorname{sech}^2(x) \tag{45}$$

The **addition identities**:

The corresponding formulae for trigonometric functions are usually known as the 'addition formulae'.

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \tag{46}$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \tag{47}$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} \tag{48}$$

The **double-argument identities**:

The corresponding formulae for trigonometric functions are usually known as the 'double-argument formulae'.

$$\sinh(2x) = 2\sinh(x)\cosh(x) \tag{49}$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) \tag{50}$$

$$\cosh(2x) = 1 + 2\sinh^2(x) \tag{51}$$

$$\cosh(2x) = 2\cosh^2(x) - 1 \tag{52}$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)} \tag{53}$$

The **half-argument identities**:

The corresponding formulae for trigonometric functions are usually known as the 'half-argument formulae'.

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}[1 + \cosh(x)] \tag{54}$$

$$\sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}[-1 + \cosh(x)] \tag{55}$$

and if $t = \tanh(x/2)$ then

$$\sinh(x) = \frac{2t}{1-t^2} \tag{56}$$

$$\cosh(x) = \frac{1+t^2}{1-t^2} \tag{57}$$

$$\tanh(x) = \frac{2t}{1+t^2} \tag{58}$$

The **sum identities**:

$$\sinh(x) + \sinh(y) = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \tag{59}$$

$$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \tag{60}$$

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$$\cosh(x)+\cosh(y)=2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right) \quad (61)$$

$$\cosh(x)-\cosh(y)=2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \quad (62)$$

product identities

The **product identities:**

$$2\sinh(x)\cosh(y)=\sinh(x+y)+\sinh(x-y) \quad (63)$$

$$2\cosh(x)\cosh(y)=\cosh(x+y)+\cosh(x-y) \quad (64)$$

$$2\sinh(x)\sinh(y)=\cosh(x+y)-\cosh(x-y) \quad (65)$$

The identities can be 'derived' quite simply from their trigonometric counterparts using Osborne's rules: change sin to sinh, cos to cosh, etc. in the trigonometric identities and then whenever sin × sin occurs change the sign (i.e. in $\sin^2$, $\tan^2$, cosec^2 and $\cot^2$).

All of these identities can be derived from the definitions of sinh (x) and cosh (x) in terms of e^x as the following questions demonstrate.

◆ Use the definitions of sinh (x) and cosh (x) in terms of e^x to show that sinh $(x+y)$ = sinh (x) cosh (y) + cosh (x) sinh (y).

◇ The left-hand side can be written as

$$\frac{1}{2}(e^{x+y}-e^{-x-y})$$

The right-hand side seems complicated but reduces to the left-hand side after some algebraic manipulation:

$$\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)$$

$$=\frac{1}{4}[(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y})$$

$$+(e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y})]=\frac{1}{2}(e^{x+y}-e^{-x-y}) \quad \square$$

Question T11 Use the definition of the hyperbolic functions in terms of the exponential function to show that

$$\sinh(2x)=2\sinh(x)\cosh(x) \quad \square$$

In many cases it is more convenient to derive identities from other identities rather than from the basic definitions. For example, by putting $x=y$ in

$$\sinh(x+y)=\sinh(x)\cosh(y)+\cosh(x)\sinh(y)$$

we obtain $\sinh(2x)=2\sinh(x)\cosh(x)$

This is certainly more straightforward than using the basic definitions; the other double-argument identities follow from the addition identities in a similar fashion.

Question T12 Use the addition identities to show that

$$\cosh(2x)=2\cosh^2(x)-1 \quad \text{and} \quad \tanh(2x)=\frac{2\tanh(x)}{1+\tanh^2(x)}$$

5 Differentiating hyperbolic functions

Study comment This section contains some results that are important in their own right, but its main purpose is to give you further practice in using the techniques of differentiation.

As shown in Question T3, the derivatives of the basic hyperbolic functions are given by

$$\frac{d}{dx}\sinh(x)=\cosh(x) \quad \text{(Eqn 17)}$$

$$\frac{d}{dx}\cosh(x)=\sinh(x) \quad \text{(Eqn 18)}$$

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These important results can be used to find the derivatives of other hyperbolic functions. Here we consider some specific examples.

Example 2 Show that $\frac{d^2}{dx^2}\sinh(3x) = 9\sinh(3x)$

Solution $\frac{d}{dx}\sinh(3x) = 3\cosh(3x)$ and $\frac{d}{dx}3\cosh(3x) = 9\sinh(3x)$ □

To justify the first step, let $y = 3x$ and use the chain rule to write

$$\frac{d}{dx}\sinh(3x) = \frac{dy}{dx}\frac{d}{dy}\sinh(y)$$
$$= 3\cosh(y)$$
$$= 3\cosh(3x)$$

Example 3 Show that $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

Solution We know that $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

and using the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$

we have $\frac{d}{dx}\tanh(x) = \frac{1}{\cosh(x)} \times \cosh(x) - \frac{\sinh(x)}{\cosh^2(x)} \times \sinh(x)$

so $\frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x)$ □

Question T13 Use Equations 17 and 18 to show that

$$\frac{d}{dx}\coth(x) = -\operatorname{cosech}^2(x)$$ □

Example 4 Show that $\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{1}{\sqrt{x^2+1}}$

This result is of great importance in *integration*, as discussed elsewhere in *FLAP*.

Solution There are two approaches:

(a) Let $y = \operatorname{arcsinh}(x)$ so that we have to find $\frac{dy}{dx}$.

We have $x = \sinh(y)$, and differentiating both sides of this equation with respect to x gives us

$$1 = \frac{d}{dx}[\sinh(y)] = \frac{dy}{dx}\frac{d}{dy}[\sinh(y)] \quad \text{(from the chain rule)}$$

and therefore $1 = \frac{dy}{dx}\cosh(y)$

which gives us $\frac{dy}{dx} = \frac{1}{\cosh(y)}$

We need the derivative in terms of x, so we use the identity of Equation 43 to write

$$\cosh(y) = \sqrt{1 + \sinh^2(y)} = \sqrt{1 + x^2}$$

(Notice that the positive square root is taken in this expression because $\cosh(y) \geq 1$.)

We can now see that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ and therefore $\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{1}{\sqrt{x^2+1}}$

(b) Differentiating the identity $\operatorname{arcsin}(x) = \log_e\left(x + \sqrt{x^2+1}\right)$

with respect to x, and using the fact that $\frac{d}{dx}\log_e(f(x)) = \frac{f'(x)}{f(x)}$,

we have

$$\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{d}{dx}\log_e\left(x + \sqrt{x^2+1}\right)$$
$$= \frac{1}{x + \sqrt{x^2+1}}\left(1 + 2x \times \frac{1}{2} \times \frac{1}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$ □

Here we use the standard notation that $f'(a)$ is $\frac{df}{dx}$ evaluated at $x = a$, i.e. $f'(a) = \left[\frac{df(x)}{dx}\right]_{x=a}$

Question T14 Show that $\frac{d}{dx}\operatorname{arctanh}(x) = \frac{1}{1-x^2}$ by using:

(a) the known derivative of $\tanh(x)$;

(b) the identity for $\operatorname{arctanh}(x)$ in terms of the logarithmic function. □

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Example 5 Find $\frac{d}{dx}\operatorname{sech}(x)$

Solution $\frac{d}{dx}\operatorname{sech}(x) = \frac{d}{dx}[\cosh(x)]^{-1} = -[\cosh(x)]^{-2}\frac{d}{dx}\cosh(x)$

$$= -\frac{\sinh(x)}{\cosh^2(x)} = -\operatorname{sech}(x)\tanh(x)$$ □

Question T15 Find $\frac{d}{dx}\operatorname{cosech}(x)$ □

Example 6 Find $\frac{d}{dx}\sinh^3(2x)$

Solution $\frac{d}{dx}\sinh^3(2x) = 3\sinh^2(2x)\frac{d}{dx}\sinh(2x)$

$$= 6\sinh^2(2x)\cosh^2(2x) = 3\sinh(2x)\sinh(4x)$$ □

Question T16 Find $\frac{d}{dx}\cosh^5(3x)$ □

Example 7 Find $\frac{d}{dx}[\sinh(x)\tanh(x)]$

Solution $\frac{d}{dx}[\sinh(x)\tanh(x)] = \frac{d}{dx}\sinh(x)\tanh(x) + \sinh(x)\frac{d}{dx}\tanh(x)$

$$= \cosh(x)\tanh(x) + \sinh(x)\operatorname{sech}^2(x)$$

$$= \sinh(x)[1 + \operatorname{sech}^2(x)]$$ □

Question T17 Find $\frac{d}{dx}[\cosh(x)\coth(x)]$ □

Example 8 Find $\frac{d}{dx}\sinh(3x^2 + 1)$

Solution $\frac{d}{dx}\sinh(3x^2 + 1) = \cosh(3x^2 + 1)\frac{d}{dx}(3x^2 + 1)$

$$= 6x\cosh(3x^2 + 1)$$ □

Question T18 Find $\frac{d}{dx}\cosh(5x^4 + 1)$ □

Example 9 Find $\frac{d}{dx}\left(\exp[\cosh(5x)]\right)$

Solution $\frac{d}{dx}\left(\exp[\cosh(5x)]\right) = \exp[\cosh(5x)]\frac{d}{dx}[\cosh(5x)]$

$$= 5\exp[\cosh(5x)]\sinh(5x)$$ □

Question T19 Find $\frac{d}{dx}\left(\exp[\sinh^2(x)]\right)$ □

Example 10 Find $\frac{d}{dx}\log_e[\cosh(x)]$

Solution $\frac{d}{dx}\log_e[\cosh(x)] = \frac{1}{\cosh(x)} \times \frac{d}{dx}\cosh(x)$

$$= \frac{\sinh(x)}{\cosh(x)} = \tanh(x)$$ □

Question T20 Find $\frac{d}{dx}\log_e[\sinh(x)]$ □

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Example 11 A heavy cable is suspended between two pylons of the same height, as shown schematically in Figure 16. It can be shown that the y-coordinate of the cable satisfies the (differential) equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

where T_0 is the tension in the cable at $x = 0$, ρ is the density per unit length of the cable, and g is the magnitude of the acceleration due to gravity. Show that

$$y = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right)$$

is a solution to this equation.

This is the example mentioned in the *Introduction* (Figure 1).

The shape of the curve taken up by a cable suspended between two points is known as a *catenary* (from the Latin *catena*, a chain).

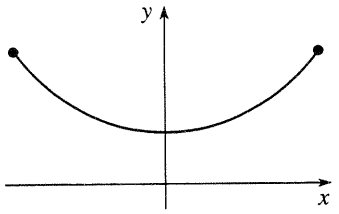


Figure 16 A heavy cable suspended between two points (such as two pylons).

Solution If $y = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right)$ then $\frac{dy}{dx} = \sinh\left(\frac{\rho g x}{T_0}\right)$

and $\frac{d^2y}{dx^2} = \frac{\rho g}{T_0}\cosh\left(\frac{\rho g x}{T_0}\right)$ which is the left-hand side of the differential equation.

Using the identity $\cosh^2(x) - \sinh^2(x) = 1$

we see that the right-hand side of the differential equation can be written as

$$\frac{\rho g}{T_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T_0}\sqrt{1+\sinh^2\left(\frac{\rho g x}{T_0}\right)} = \frac{\rho g}{T_0}\cosh\left(\frac{\rho g x}{T_0}\right)$$

which verifies that $y = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right)$ is indeed a solution. □

6 Closing items

6.1 Module summary

1 There are *hyperbolic functions* corresponding to all the trigonometric functions, and though they are similar in some respects they are very different in others. In particular, the hyperbolic functions are not periodic, and, $\sinh(x)$ and $\cosh(x)$ are not constrained to lie between -1 and 1. Hyperbolic functions are named by appending 'h' to the corresponding trigonometric function.

2 $\sinh(x)$ and $\cosh(x)$ are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{(Eqn 3)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{(Eqn 4)}$$

3 $\tanh(x)$ is defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{(Eqn 5)}$$

4 A hyperbola can be defined by means of the parametric equations

$$x = a\cosh(s) \qquad \text{(Eqn 6)}$$

$$y = b\sinh(s) \qquad \text{(Eqn 7)}$$

5 The Taylor expansions of $\sinh(x)$ and $\cosh(x)$ about $x = 0$ are

$$\sinh(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \qquad \text{(Eqn 11)}$$

$$\cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \qquad \text{(Eqn 13)}$$

6 The first few terms of the Taylor expansion of $\tanh(x)$ about $x = 0$ are

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots \qquad \text{(Eqn 15)}$$

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7 The derivatives of the two basic hyperbolic functions are

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \text{(Eqn 17)}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \text{(Eqn 18)}$$

8 The trigonometric and hyperbolic functions are related by

$$\sinh(ix) = i\sin(x) \qquad \text{(Eqn 22)}$$

$$\cosh(ix) = \cos(x) \qquad \text{(Eqn 23)}$$

$$\sin(ix) = i\sinh(x)$$

$$\cos(ix) = \cosh(x)$$

9 *Reciprocal hyperbolic functions* are defined by

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \text{provided } x \neq 0 \qquad \text{(Eqn 24)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \text{for all } x \qquad \text{(Eqn 25)}$$

$$\coth(x) = \frac{1}{\tanh(x)} \quad \text{provided } x \neq 0 \qquad \text{(Eqn 26)}$$

10 *Inverse hyperbolic functions* are defined by

$$\operatorname{arcsinh}(\sinh(x)) = x \qquad \text{(Eqn 30)}$$

$$\operatorname{arccosh}(\cosh(x)) = x \quad \text{for } x \geq 0 \qquad \text{(Eqn 32b)}$$

$$\operatorname{arctanh}(\tanh(x)) = x \qquad \text{(Eqn 31)}$$

11 The inverse hyperbolic functions can be written in terms of the logarithmic function

$$\operatorname{arcsinh}(x) = \log_e\left(x + \sqrt{x^2+1}\right) \qquad \text{(Eqn 36)}$$

$$\operatorname{arccosh}(x) = \log_e\left(x + \sqrt{x^2-1}\right) \quad \text{for } x \geq 1 \qquad \text{(Eqn 37)}$$

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1 \qquad \text{(Eqn 38)}$$

12 There are a large number of *hyperbolic function identities*, such as

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \text{(Eqn 43)}$$

The most important are listed in Section 4.

6.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of this module.

A2 Relate the trigonometric and hyperbolic functions by means of the complex number, *i*.

A3 Sketch graphs for the more common hyperbolic functions

A4 Recognize the more common identities for the hyperbolic functions and use them to solve mathematical and physical problems.

A5 Differentiate the hyperbolic functions.

A6 Derive, identify and use the Taylor series for the hyperbolic functions.

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6.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2*) Use the definitions of trigonometric and hyperbolic functions in terms of the exponential function to show that: (a) $\tanh(x) = -i\tan(ix)$, (b) $\operatorname{cosech}(x) = i\operatorname{cosec}(ix)$.

Question E2 (*A4*) Use the double-argument identities to show that

(a) $\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}[1+\cosh(x)]$, (b) $\sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}[-1+\cosh(x)]$.

Question E3 (*A5*) Show that $y(t) = A\sinh(\omega t) + B\cosh(\omega t)$

is a solution of $\frac{d^2y(t)}{dt^2} = \omega^2 t$. If $y(0) = 4$ and $y'(0) = 1$, find A and B.

Question E4 (*A4*) Suppose that we are given the values of A and B. Use the identities given in Section 4 to find values of C and y (in terms of A and B) such that

$$A\sinh(x) + B\cosh(x) = C\sinh(x+y)$$

Explain any restrictions on the possible values for A and B.

Question E5 (*A3*) From the graphs given in this module, which hyperbolic functions would you expect to have a domain which excludes the point $x = 0$. Confirm your suspicions by looking at algebraic expressions for the functions that you have picked out. (You should include the reciprocal and inverse hyperbolic functions in your discussion.)

Question E6 (*A4*) Starting from the definition of $\sinh(x)$ in terms of the exponential function, show that

$$\operatorname{arcsinh}(x) = \log_e\left(x + \sqrt{x^2+1}\right)$$

Question E7 (*A5*) Use the known derivative of $\tanh(x)$ to express $\frac{d}{dx}\log_e[\tanh(x)]$ in terms of hyperbolic functions.

Question E8 (*A5*) Find $\frac{d}{dx}\tanh^3(2x)$.

Question E9 (*A4* and *A6*) Use the series for $\sinh(x)$ and $\cosh(x)$ to find a first estimate for a solution of the equation

$$\sinh(x) = 0.01\cosh(x)$$

then use the logarithmic form of $\tanh(x)$ to find a more accurate value for the solution.

Study comment Now that you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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7 Answers and comments

Fast track answers

F1 $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

and $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

F2 $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$, and $\coth(x) = \dfrac{1}{\tanh(x)}$

provided $x \neq 0$.

Using the expression for cosh (x) given in Answer F1 we have

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

However $\sec(x) = \dfrac{1}{\cos(x)} = \dfrac{2}{e^{ix} + e^{-ix}}$ and therefore

$$\sec(ix) = \frac{2}{e^{-x} + e^{x}} = \operatorname{sech}(x).$$

In a similar way, using the expressions for sinh (x) and cosh (x) given in Answer F1 we have

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

However $\cot(x) = \dfrac{\cos(x)}{\sin(x)} = i\dfrac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$

and therefore

$$i\cot(ix) = i \times i\frac{e^{-x} + e^{x}}{e^{-x} - e^{x}} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \coth(x)$$

F3 Let $y = \operatorname{arccoth}(x)$ then

$$x = \coth(y) = \frac{\cosh(y)}{\sinh(y)} = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

which can be rearranged as follows:

$$xe^y - xe^{-y} = e^y + e^{-y}$$

$$e^y(x-1) = e^{-y}(x+1)$$

$$e^{2y} = \left(\frac{x+1}{x-1}\right)$$

Taking the logarithm of both sides gives us

$$2y = \log_e\left(\frac{x+1}{x-1}\right)$$

and therefore $\operatorname{arccoth}(x) = y = \dfrac{1}{2}\log_e\left(\dfrac{x+1}{x-1}\right)$

The condition $|x| > 1$ is necessary so that the argument of the logarithmic function is positive (and this is consistent with Figure 14).

$$\operatorname{arccoth}(1/s) = \frac{1}{2}\log_e\left(\frac{(1/s)+1}{(1/s)-1}\right) = \frac{1}{2}\log_e\left(\frac{1+s}{1-s}\right)$$

$$= \frac{1}{2}[\log_e(1+s) - \log_e(1-s)]$$

$$= \frac{1}{2}\left[\left(s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \ldots\right) + \left(s + \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{4} + \ldots\right)\right]$$

$$= s + \frac{s^3}{3} + \frac{s^5}{5} + \ldots$$

which is valid for $|s| < 1$ (since the series for $\log_e(1 - s)$ and $\log_e(1 + s)$ are valid for $|s| < 1$). From this series (putting $s = 0.1$) we have

$$\operatorname{arccoth}(10) \approx 0.1 + \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5} \approx 0.10033533$$

From my calculator

$$\operatorname{arccoth}(10) = \frac{1}{2}\log_e\left(\frac{10+1}{10-1}\right) = \frac{1}{2}\log_e(11/9)$$

$$\approx 0.10033535$$

F4 $\dfrac{d}{dx}\operatorname{arccoth}(x) = \dfrac{d}{dx}\left[\dfrac{1}{2}\log_e\left(\dfrac{x+1}{x-1}\right)\right]$

$$= \frac{1}{2}\frac{d}{dx}[\log_e(x+1) - \log_e(x-1)]$$

$$= \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) = \frac{1}{1-x^2}$$

Ready to study answers

R1 (a) $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(b) Using the general formula for the solution of a quadratic equation we have $x = \dfrac{a \pm \sqrt{a^2 - 4}}{2}$ and since $a \geq 2$ we see that both roots will be real. The solutions of the equation $e^{2w} - ae^w + 1 = 0$ are

$$w = \log_e(a \pm \sqrt{a^2 - 4}) - \log_e 2$$

Consult *Taylor expansion* in the *Glossary* for further information.

R2 (a) $\arcsin(1/3) \approx 0.3398$ and $\dfrac{1}{\sin(1/3)} \approx 3.06$

Notice that these results are *not* the same. Whereas arcsin (x) is the inverse of the sine function, $\dfrac{1}{\sin(x)}$ is the reciprocal of the sine function.

(b) $\sin[\sin(1/3)] \approx 0.321$ and $\sin^2(1/3) \approx 0.107$

Notice again that these results are not the same. This is because the expression

$\sin[\sin(x)]$

means 'take the sine of x and then take the sine of the result'.

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By contrast

$\sin^2(x)$

means $\sin(x) \times \sin(x)$, i.e. the square of $\sin(x)$.

Consult *trigonometric functions* in the *Glossary* for further information.

R3 (a) We have

$$s(x) \times s(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = -\frac{1}{4}(e^{2ix} - 2 + e^{-2ix})$$

and $$c(x) \times c(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{1}{4}(e^{2ix} + 2 + e^{-2ix})$$

Adding these two results gives

$$s(x) \times s(x) + c(x) \times c(x) = 1$$

(b) From *Euler's formula* $e^{ix} = \cos(x) + i\sin(x)$

we have

$$s(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \frac{\cos(x) + i\sin(x) - [\cos(x) - i\sin(x)]}{2i} = \sin(x)$$

and

$$c(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$= \frac{\cos(x) + i\sin(x) + [\cos(x) - i\sin(x)]}{2} = \cos(x)$$

Consult *Euler's formula* in the *Glossary* for further information.

R4 $$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{1}{v(x)}\frac{du(x)}{dx} - \frac{u(x)}{v(x)^2}\frac{dv(x)}{dx}$$

$$= \frac{v(x)\dfrac{du(x)}{dx} - u(x)\dfrac{dv(x)}{dx}}{v(x)^2}$$

and $$\frac{d}{dx}g(f(x)) = \frac{dg}{df} \times \frac{df}{dx}$$

Consult *quotient rule* and *chain rule* in the *Glossary* for further information.

Text answers

T1 (a) $\sinh(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\dfrac{e^{x} - e^{-x}}{2} = -\sinh(x)$,

therefore it is an *odd* function

and $\cosh(-x) = \dfrac{e^{-x} + e^{x}}{2} = \cosh(x)$, therefore it is an *even* function.

(b) Adding the expressions for $\sinh(x)$ and $\cosh(x)$

$$\sinh(x) + \cosh(x) = \frac{e^{x} - e^{-x}}{2} + \frac{e^{x} + e^{-x}}{2} = e^{x}$$

(c) $$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$$

T2 $$v = \sqrt{A\lambda \tanh\left(\frac{6.3d}{\lambda}\right)}$$

$$= \sqrt{(1.8\,\mathrm{m\,s^{-2}}) \times 18.9\,\mathrm{m} \times \tanh\left(\frac{6.3 \times 6\,\mathrm{m}}{18.9\,\mathrm{m}}\right)} \approx 5.73\,\mathrm{m\,s^{-1}}$$

T3 $$\frac{d}{dx}\sinh(x) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{d}{dx}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(2n+1)x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} = \cosh(x)$$

Also $$\frac{d}{dx}\cosh(x) = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\frac{d}{dx}x^{2n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{(2n)!}2nx^{2n-1} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)!}x^{2n-1}$$

and since there is no $n = 0$ term in this series, it is convenient to change the summation variable by setting $m = n - 1$. At the lower limit $n = 1$ corresponds to $m = 0$, and therefore

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}x^{2n-1} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}x^{2m+1} = \sinh(x)$$

T4 (a) $$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = -\frac{1}{i}\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= -\frac{1}{i}\sinh(x)$$

implying that $\sin(ix) = i\sinh(x)$

(b) $$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh(x)$$

implying that $\cos(ix) = \cosh(x)$

T5 Since the series for $\cos(x)$ and $\cosh(x)$ are

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$\cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

we have $$\cos(ix) = \sum_{n=0}^{\infty}\frac{(-1)^n (ix)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = \cosh(x)$$

T6 Using the series for $\tanh(x)$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots$$

we obtain

$$\tanh(0.4) = 0.4 - \frac{(0.4)^3}{3} + \frac{2(0.4)^5}{15} - \frac{17(0.4)^7}{315} + \ldots$$

$$= 0.4 - 0.021\,333 + 0.001\,365 - 0.000\,088 \ldots$$

$$\approx 0.380$$

To check the result on a calculator (which doesn't have a tanh button) we can use

$$\tanh(0.4) = \frac{e^{0.4} - e^{-0.4}}{e^{0.4} + e^{-0.4}} \approx 0.379\,949 \approx 0.380$$

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T7 Dividing the identity $\cosh^2(x) - \sinh^2(x) = 1$, given in Question T1(c), by $\sinh^2(x)$ gives

$$\frac{\cosh^2(x)}{\sinh^2(x)} - 1 = \frac{1}{\sinh^2(x)}$$

and therefore

$\coth^2(x) - 1 = \operatorname{cosech}^2(x)$, i.e. $\coth^2(x) - \operatorname{cosech}^2(x) = 1$.

T8 (a) Let $x = \operatorname{cosech}(y)$ then $\sinh(y) = 1/x$

and therefore $y = \operatorname{arcsinh}(1/x)$.

However, from the definition of the arccosech function, we have

$\operatorname{arccosech}(x) = y = \operatorname{arcsinh}(1/x)$

(b) Let $x = \operatorname{sech}(y)$ then $\cosh(y) = 1/x$ so that $y = \operatorname{arccosh}(1/x)$.

However, from the definition of the arcsech function, we have

$\operatorname{arcsech}(x) = y = \operatorname{arccosh}(1/x)$

(c) Let $x = \coth(y)$ then $\tanh(y) = 1/x$ so that $y = \operatorname{arctanh}(1/x)$.

However, from the definition of the arccoth function, we have

$\operatorname{arccoth}(x) = y = \operatorname{arctanh}(1/x)$

T9 Let $y = \operatorname{arctanh}(x)$

then $$x = \tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

and therefore $e^y(1 - x) = e^{-y}(1 + x)$

which gives $$e^{2y} = \left(\frac{1+x}{1-x}\right)$$

The condition $\left(\frac{1+x}{1-x}\right) = e^{2y} > 0$ ensures that $-1 < x < 1$,

and taking logarithms of both sides gives $2y = \log_e\left(\frac{1+x}{1-x}\right)$,

which gives the required result.

T10 Using the identity

$\cosh^2(x) - \sinh^2(x) = 1$

we can write the equation as

$(\sinh^2(x) + 1) + \sinh(x) = 3$

which can be rearranged to give

$\sinh^2(x) + \sinh(x) - 2 = 0$

and then factorized to read

$(\sinh(x) - 1)(\sinh(x) + 2) = 0$

This equation has the solutions

$x = \operatorname{arcsinh}(1)$ and $x = \operatorname{arcsinh}(-2)$

Using $\operatorname{arcsinh}(x) = \log_e(x + \sqrt{x^2 + 1})$

the two solutions can be written as

$x = \log_e(1 + \sqrt{2})$ and $x = \log_e(\sqrt{5} - 2)$

T11 $$2\sinh(x)\cosh(x) = 2 \times \left(\frac{e^x - e^{-x}}{2}\right) \times \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x)$$

T12 (a) If we set $y = x$ in the identity

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

we find $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

But we also have the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

and so, substituting for $\sinh(x)$ we obtain

$\cosh(2x) = 2\cosh^2(x) - 1$

(b) Setting $y = x$ in the identity

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

leads directly to

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

T13 $$\frac{d}{dx}\coth(x) = \frac{d}{dx}\left[\frac{\cosh(x)}{\sinh(x)}\right]$$

$$= \frac{1}{\sinh(x)}\frac{d}{dx}\cosh(x) - \frac{\cosh(x)}{\sinh^2(x)}\frac{d}{dx}\sinh(x)$$

$$= \frac{1}{\sinh(x)} \times \sinh(x) - \frac{\cosh(x)}{\sinh^2(x)} \times \cosh(x)$$

$$= \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = -\frac{1}{\sinh^2(x)} = -\operatorname{cosech}^2(x)$$

T14 (a) Setting $x = \tanh(y)$ and differentiating both sides with respect to x we find

$$1 = \frac{dy}{dx}\frac{d}{dy}\tanh(y) = \frac{dy}{dx}\operatorname{sech}^2(y)$$

and therefore $$\frac{dy}{dx} = \frac{d}{dx}\arctan(x) = \frac{1}{\operatorname{sech}^2(y)}$$

Using the identity $\operatorname{sech}^2(y) + \tanh^2(y) = 1$ gives us

$$\operatorname{sech}(y) = \sqrt{1 - \tanh^2(y)} = \sqrt{1 - x^2}$$

and therefore $$\frac{d}{dx}\operatorname{arctanh}(x) = \frac{1}{1 - x^2}$$

(b) Differentiating the identity

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = \frac{1}{2}[\log_e(1 + x) - \log_e(1 - x)]$$

with respect to x gives us

$$\frac{d}{dx}\operatorname{arctanh}(x) = \frac{1}{2(1+x)} + \frac{1}{2(1-x)} = \frac{1}{1 - x^2}$$

T15 $$\frac{d}{dx}\operatorname{cosech}(x) = \frac{d}{dx}[\sinh(x)]^{-1}$$

$$= -[\sinh(x)]^{-2}\frac{d}{dx}\sinh(x)$$

$$= -\frac{\cosh(x)}{\sinh^2(x)} = -\operatorname{cosech}(x)\coth(x)$$

T16 $$\frac{d}{dx}\cosh^5(3x) = 5\cosh^4(3x)\frac{d}{dx}\cosh(3x)$$

$$= 15\cosh^4(3x)\sinh(3x)$$

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T17 $\dfrac{d}{dx}[\cosh(x)\coth(x)]$

$$= \frac{d\cosh(x)}{dx}\coth(x) + \cosh(x)\frac{d\coth(x)}{dx}$$

$$= \sinh(x)\coth(x) - \cosh(x)\,\mathrm{cosech}^2(x)$$

$$= \cosh(x)\,(1 - \mathrm{cosech}^2(x))$$

T18 $\dfrac{d}{dx}\cosh(5x^4+1) = \sinh(5x^4+1)\dfrac{d}{dx}(5x^4+1)$

$$= 20x^3\sinh(5x^4+1)$$

T19 $\dfrac{d}{dx}[\exp(\sinh^2(x))] = \exp(\sinh^2(x))\dfrac{d}{dx}\sinh^2(x)$

$$= 2\sinh(x)\cosh(x)\exp(\sinh^2(x))$$

$$= \sinh(2x)\exp(\sinh^2(x))$$

T20 $\dfrac{d}{dx}\log[\sinh(x)] = \dfrac{1}{\sinh(x)}\dfrac{d}{dx}\sinh(x)$

$$= \frac{\cosh(x)}{\sinh(x)} = \coth(x)$$

Exit test answers

E1 (a) We can write $\tanh(x)$ as $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

and $\tan(x)$ as $\tan(x) = \dfrac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$ which means that

$$-i\tan(ix) = -\frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \tanh(x)$$

(b) Using the definitions of cosech (x) and cosec (x), we can write

$$\mathrm{cosech}(x) = \frac{2}{e^x - e^{-x}} \quad \text{and} \quad \mathrm{cosec}(x) = \frac{2i}{e^{ix} - e^{-ix}}$$

so that $i\,\mathrm{cosec}(ix) = \dfrac{-2}{e^{-x} - e^{x}} = \dfrac{2}{e^x - e^{-x}} = \mathrm{cosech}(x)$

(Reread Subsections 2.1 and 2.3 if you had difficulty with this question.)

E2 (a) Using the identity $\cosh(2x) = 2\cosh^2(x) - 1$ we find

$$\cosh^2(x) = \frac{1}{2}[1 + \cosh(2x)]$$

and therefore (replacing x by $x/2$)

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}[1 + \cosh(x)]$$

(b) In a similar way, using the identity

$$\cosh(2x) = 1 + 2\sinh^2(x)$$

gives us $\sinh^2(x) = \dfrac{1}{2}[-1 + \cosh(2x)]$

and therefore $\sinh^2\left(\dfrac{x}{2}\right) = \dfrac{1}{2}[-1 + \cosh(x)]$.

(Reread Section 4 if you had difficulty with this question.)

E3 Using the standard results

$$\frac{d}{dt}\sinh(\omega t) = \omega\cosh(\omega t) \quad \text{and} \quad \frac{d}{dt}\cosh(\omega t) = \omega\sinh(\omega t),$$

we obtain $\dfrac{d}{dt}y(t) = \omega[A\cosh(\omega t) + B\sinh(\omega t)]$

and therefore

$$\frac{d^2}{dt^2}y(t) = \omega^2[A\sinh(\omega t) + B\cosh(\omega t)] = \omega^2 y(t)$$

If we set $y(0) = 4$ we find

$4 = A\sinh(0) + B\cosh(0) = B$ and therefore $B = 4$.

If we set $y'(0) = 1$, we find

$1 = \omega[A\cosh(0) + B\sinh(0)] = \omega A$

which implies that $A = 1/\omega$.

(Reread Section 5 if you had difficulty with this question.)

E4 Using the identity

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

we obtain $\dfrac{A}{C} = \cosh(y)$ and $\dfrac{B}{C} = \sinh(y)$

then using the identity $\cosh^2(y) - \sinh^2(y) = 1$ we obtain

$$A^2 - B^2 = C^2$$

which is only possible if $|B| \leq |A|$. We then have $C = \pm\sqrt{A^2 - B^2}$ with an apparent ambiguity in the sign of C. However, we know that $\dfrac{A}{C} = \cosh(y) \geq 1$ and therefore A and C must have the same sign. Finally, $y = \mathrm{arctanh}\,(B/A)$, which is well-defined if $|B| < |A|$.

The conclusion is that $A\sinh(x) + B\cosh(x)$ can be written in the form

$C\sinh(x+y)$ with $C = \pm\sqrt{A^2 - B^2}$ and $y = \mathrm{arctanh}\,(B/A)$ provided that $|B| < |A|$.

(Reread Section 4 if you had difficulty with this question.)

E5 The functions are:

(a) coth (x) (see Figure 8), since $\coth(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ which, on setting $x = 0$, becomes $\dfrac{1+1}{1-1}$ which is undefined.

(b) cosech (x) (see Figure 6), since cosech $(x) = \dfrac{2}{e^x - e^{-x}}$ which, on setting $x = 0$, becomes $\dfrac{2}{1-1}$ and this is again undefined.

(c) arccosh (x) (see Figure 11), is not defined at $x = 0$ (no value of x makes cosh (x) equal to zero).

(d) arccosech (x) (see Figure 12), is not defined at $x = 0$ (no value of x makes cosech (x) equal to zero).

(e) arcsech (x) (see Figure 13), is not defined at $x = 0$ (no value of x makes sech (x) equal to zero).

(f) arccoth (x) (see Figure 14), is not defined at $x = 0$ (no value of x makes coth (x) equal to zero).

(Reread Section 3 if you had difficulty with this question.)

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E6 Let $y = \text{arcsinh}\,(x)$ then $x = \sinh\,(y) = \dfrac{e^y - e^{-y}}{2}$

which can be rearranged as $e^{2y} - 2xe^y - 1 = 0$.

This is a quadratic equation in e^y and its two solutions are $e^y = x \pm \sqrt{x^2 + 1}$ We can reject the negative root because $e^y \geq 0$ would imply $x - \sqrt{x^2 + 1} > 0$, which would give $x^2 > x^2 + 1$, which is impossible. It follows that only the positive root is relevant, and taking the logarithm of both sides of the equation gives us, $\text{arcsinh}\,(x) = \log_e(x + \sqrt{x^2 + 1})$.

(Reread Subsection 3.2 if you had difficulty with this question.)

E7 See Example 3 for the derivation of the result

$$\frac{d}{dx}\tanh(x) = \text{sech}^2\,(x)$$

We can use this to obtain

$$\frac{d}{dx}\log_e[\tanh(x)] = \frac{1}{\tanh(x)}\frac{d}{dx}\tanh(x)$$

$$= \frac{\text{sech}^2\,(x)}{\tanh(x)} = \frac{1}{\cosh(x)\sinh(x)}$$

$$= \frac{2}{\sinh(2x)} = 2\,\text{cosech}\,(2x)$$

(Reread Section 5 if you had difficulty with this question.)

E8 $\dfrac{d}{dx}\tanh^3(2x) = 3\tanh^2(2x)\dfrac{d}{dx}\tanh(2x)$

$$= 6\tanh^2(2x)\,\text{sech}^2(2x)$$

(Reread Section 5 if you had difficulty with this question.)

E9 From Equations 11 and 13 we have

$$\sinh x \approx x + \frac{x^3}{6} \quad \text{and} \quad \cosh x \approx 1 + \frac{x^2}{2}$$

so that the given equation becomes

$$x + \frac{x^3}{6} \approx (0.01)\left(1 + \frac{x^2}{2}\right)$$

Ignoring the terms in x^2 and x^3 we obtain the rough estimate $x \approx 0.01$ for the solution. We can obtain a better estimate for the solution if we observe that the given equation implies, $\tanh(x) = 0.01$, so that using Equation 38 we have

$$\text{arctanh}\,(x) = \frac{1}{2}\log_e\left(\frac{1 + 0.01}{1 - 0.01}\right) \approx 0.0100003.$$

(Reread Subsections 2.2, 2.4 and 3.2 if you had difficulty with this question.)

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