Further differentiation

Module M4.3

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1 Opening items

1.1 Module introduction

Applications of differentiation are widespread throughout physics. For example, given the displacement from the origin of a moving object as a function of time, the corresponding velocity can be found by differentiating that function. Or, given a simple electrical circuit consisting of a resistor and an inductor connected in series with a charged capacitor and a switch (i.e. an *LCR* circuit), the potential differences across the inductor and resistor are, respectively, $L\frac{dI}{dt}(t)$ and $RI(t)$ where $I(t)$ is the current in the circuit at time t after the switch is closed and L and R are constants. It is also the case that in such a circuit $I(t) = \frac{dq}{dt}(t)$ where $q(t)$ is the charge on the capacitor at time t. Because the need to differentiate arises so often in physics it is essential that you should be able to differentiate readily a wide variety of functions. The techniques of basic differentiation should enable you to differentiate certain standard functions such as powers, trigonometric functions, exponentials and logarithms, or certain combinations of those standard functions such as sums, constant multiples, products and quotients. This module goes further in that it introduces techniques that, when combined with those of basic differentiation, will allow you to differentiate almost all the functions of a single variable that you are likely to meet.

It is often the case that a physical quantity can be represented by a function of a variable that is itself a function of another variable. For example, the displacement from equilibrium of a one-dimensional simple harmonic oscillator may be represented by $x = A\cos(\omega t + \phi)$, but $(\omega t + \phi)$ is itself a function of t, so we might write $\theta(t) = (\omega t + \phi)$, in which case we can write $x = A\cos(\theta(t))$. Such an expression makes it clear that x is a *function of a function* of t, and that to find the velocity of such an oscillator we need to be able to differentiate a function of a function of t. Section 2 of this module describes functions of a function in more detail and introduces (in Subsection 2.2) the *chain rule* for their differentiation. The chain rule is one of the most useful techniques of calculus; among other things it enables us to differentiate *implicit functions*, *inverse functions* and *parametric functions*, all of which are discussed in Subsections 2.3, 2.4 and 2.5.

If we want to find the acceleration of an object given its position then we must proceed via its velocity; the acceleration is the rate of change of the velocity, which is itself the rate of change of the position. Hence the process requires two successive differentiations. We say that the acceleration is the *second derivative* of the position of the object. Similarly, the potential difference across the inductor in the series *LCR* circuit mentioned earlier is proportional to dI/dt but $I = dq/dt$, so the potential difference is proportional to the second derivative of the charge. Clearly, *second* (and *higher*) *derivatives* are also of interest to physicists, so they form the second major theme of this module, discussed in Section 3.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 6. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

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Question F1 Find the first four derivatives of the following functions:

(a) $F(x) = 3\sin x + 4\cos x$

(b) $y = \log_e x \quad x > 0$

The alternative notations $\ln(x)$ and e^x are often used for $\log_e x$ and $\exp(x)$, respectively.

Question F2 If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, where a is a constant, find

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Question F3 Find the first and second derivatives of the following functions:

(a) $F(t) = \exp[\alpha t + \sin(\omega t)]$ with respect to t, where α and ω are constants

(b) $F(x) = 5\log_e(\sin^2 x + 1)$ with respect to x

(c) $F(x) = \arcsin(5x)$ with respect to x

You may come across the alternative notation $\sin^{-1}(x)$ for $\arcsin(x)$ However, this is not used in *FLAP*.

1.3 Ready to study?

Study comment In order to study this module you should be familiar with the following terms: *derivative*, *differentiate*, *domain*, *function* and *inverse function*. You will need to be familiar with the following specific types of function: *exponential functions*, *logarithmic functions*, *power functions*, *trigonometric functions* (*cosine*, *sine*, *tangent*, *cosec*, *sec*, *cot*) and *inverse trigonometric functions* (*arccos*, *arcsin*, *arctan*). You should also know the meaning of the term *radian* and be aware of the need to express angles in radians when using them in the *arguments* of trigonometric functions. You should be able to differentiate *powers*, *exponential functions*, *logarithmic functions* and *trigonometric functions*, and to use the *sum rule*, *constant multiple rule*, *product rule* and *quotient rule of differentiation*. If you are unsure of any of these terms then you can review them by referring to the *Glossary*, which will also indicate where in *FLAP* they are developed. The following *Ready to study questions* will let you establish whether you need to review some of these topics before working through this module.

Question R1 Which of the following formulae can be used to define a function of x? What restrictions are there on the admissible values of x?

(a) $f(x) = x^5$

(b) $y = \arcsin x$ (i.e. $\sin y = x$)

(c) $y^2 = 4x^2$

You will find the answers to these questions in Section 6.

Question R2 For each of the following, express x as a function of y:

(a) $y = 9x^3$ (b) $y = \tan x$ (c) $y = e^{2x}$ (d) $y = \log_e(3x)$

Question R3 Define the derivative of a function $f(x)$ at $x = a$.

Question R4 For each of the functions $f(x)$ below find the derivative $f'(x)$:

(a) $8x^6 + 6x^3 - 5x^2 - 2$ (b) $3\sin x + 4\cos x$

Question R5 Find dy/dx for each of the following:

(a) $y = 6e^x$ (b) $y = x^2 - 5\log_e x$

Question R6 Differentiate the following functions of x:

(a) x^2e^x (b) $x^3\sin x$ (c) $e^x\cos x$

(d) $\frac{\log_e x}{3x^2 + x}$, $x > 0$ (e) $\frac{3\cos x}{2 + \sin x}$ (f) $\frac{x}{2 + \cos x}$

A special note about $\sqrt{x}$ Generally in *FLAP* we follow the convention that $\sqrt{x}$ may be positive or negative. Thus $\sqrt{4} = \pm 2$. A consequence of this convention is that $f(x) = \sqrt{x}$ does *not* define a function since it does not associate a *unique* value of $f(x)$ with each value of x. This would be an exceptionally inconvenient convention to follow in this module so instead we adopt the convention that $\sqrt{x}$ is positive. Of course, it remains true that the square roots of x may be positive or negative, $\sqrt{x}$ or $-\sqrt{x}$, so within this convention we will call $\sqrt{x}$ the *positive* square root of x, and not simply the square root. In a similar spirit $x^{1/2} = \sqrt{x}$ will also be a positive quantity.

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2 Derivatives of composite functions

function of a function
composite function

2.1 A function of a function

In this subsection, we introduce the concept and notation of a *function of a function* by considering the example of a *simple harmonic oscillator*, such as a (frictionless) simple pendulum or a weight on an (ideal) spring. At time t the displacement x of such a simple harmonic oscillator from its equilibrium position ($x = 0$) is given by

The idea of a function of a function is also discussed in the *FLAP* module that introduces functions. See the *Glossary* for details.

Harmonic oscillations are discussed more fully in the physics strand of *FLAP*.

$$x = A\cos(\omega t + \phi) \tag{1}$$

where A is the *amplitude* of the oscillation, ω the *angular frequency* and ϕ the *phase constant*. The expression inside the brackets, $\omega t + \phi$, is called the *phase*; it is a function of time, and its value determines the stage the oscillator has reached in its cycle of motion. The oscillator returns to the same position each time the phase increases by 2π. (☞) Now, suppose that $A = 5.00\,\text{cm}$, $\omega = 0.60\,\text{s}^{-1}$ and $\phi = \pi$, and that you are asked to calculate the displacement of the oscillator at $t = 10\,\text{s}$. To answer this question you would first need to evaluate the phase when $t = 10\,\text{s}$. This is given by

The phase here is dimensionless, however, the *phase* can be interpreted as an angle in which case it is measured in *radians*, and an increase by 2π radians represents the completion of another cycle.

$$\omega t + \phi = 0.60\,\text{s}^{-1} \times 10\,\text{s} + \pi = 9.14 \quad \text{(to two decimal places)}$$

and then substitute this value into Equation 1 to find

To evaluate cos (9.14) correctly your calculator must be in the *radian mode*.

$$x = 5.00\,\text{cm} \times \cos(9.14) = -4.94\,\text{cm}$$

In this example, x is a function of the expression $(\omega t + \phi)$, which is itself a function of t. For this reason x is said to be a **function of a function** of t, or a **composite function** of t. In Subsection 2.2 we will see how to differentiate such expressions, but for the moment we concentrate on reviewing the concept of and notation for a function of a function.

The use of boxes to represent the action of functions is helpful here. Figure 1 shows, in this form, the three functions $f(t) = 5t + 1$, $f(t) = t^6$, $f(t) = \sin t$. In each case the function converts an input value (the *argument* of the function) into an output value of the function. It is an inherent property of functions that a *single* output value corresponds to each valid input.

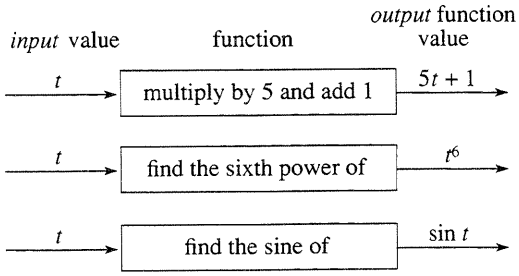


Figure 1 Box representation of a function.

For the function of a function we use two boxes in series. Figure 2 shows the action of the two composite functions $\sin(5t + 1)$ and $\cos(\omega t + \phi)$. In each case the output from the first box is used as the input for the second box.

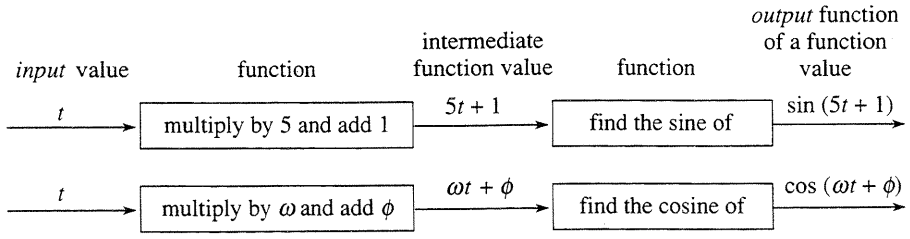


Figure 2 Composite functions.

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In general, we can consider a composite function $f(g(t))$ as depicted in Figure 3. $f(g(t))$ is read as 'f of g of t'.

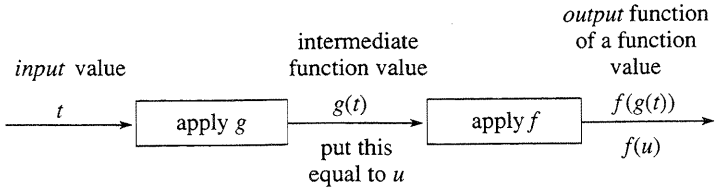


Figure 3 General composite function.

The composite function can be made more user-friendly by writing $u = g(t)$ so that the output is $f(u)$. Hence, if we consider the composite function $\sin(5t + 1)$, we can write $g(t) = 5t + 1$ so that $f(g(t)) = \sin(g(t))$ and we can denote $g(t)$ by u so that $f(u) = \sin u$ where, in this case, $u = 5t + 1$.

The notations $\sin x$ and $\sin(x)$ are identical in meaning. On the whole we will follow customary practice and omit the brackets in this module unless the argument of the function consists of something more than a single letter. Similar comments apply to other functions, such as $\tan(x)$ and $\log_e(x)$.

You will find the answers to all *Text questions* in Section 6.

Question T1 Draw a box representation of the following functions:

(a) $\cos(3t - 4)$ (b) $(t^4 + 2)^3$ □

There are two observations that you may find useful when trying to determine whether or not a function is a composite. First, decide if the function definition can be read naturally by using the word 'of'. For example, in

$$h(x) = \log_e(x^2) \quad (x \neq 0)$$

the right-hand side can be read as 'log to base e of (x squared)', which helps to identify $h(x)$ as a composite $f(g(x))$, where

$$f(x) = \log_e x$$

and $$g(x) = x^2$$

The restriction $x \neq 0$ ensures that $\log_e(x^2)$ i.e. 'f of g of x' is defined.

Sometimes the notation $(f \circ g)(x)$ is used for $f(g(x))$.

Second, think about how you would calculate function values on a calculator. For $h(x) = \log_e(x^2)$, you would proceed as follows:

first enter the value of x and square it,

then find the natural logarithm $\log_e$ of the result.

A function of a function always involves applying *first* one process, *then* another to the result of the first. Table 1 shows some examples of composite functions.

Table 1 Some composite functions.

Composite function	Read as ...	First ...	then ...
$\tan(3x)$	'$\tan$ *of* $3x$'	find $3x$	take tan (of the result)
$\log_e(1/x)$	'$\log_e$ *of* $1/x$'	find $1/x$	take $\log_e$ (of the result)
$e^{\sin x}$	'$\exp$ *of* $\sin x$'	find $\sin x$	take exp (of the result)

The following is a very important question. Make sure that you understand it before continuing.

Question T2 For each of the following functions, decide whether $F(x)$ is a composite function. If $F(x)$ is composite, express F in the form $f(g(x))$, by writing down explicit expressions for $u = g(x)$ and $f(u)$.

(a) $F(x) = e^{2x}$

(b) $F(x) = e^x \sin x$

(c) $F(x) = \log_e(x^3)$

(d) $F(x) = \sin(x^2 + 2x + 1)$

(e) $F(x) = \sin^2 x$ (Remember that $\sin^2 x$ means $(\sin x)^2$)

(f) $F(x) = \sin^3 x$

(g) $F(x) = (\cos x)^{0.5}$ □

Two points are worthy of comment to save you from possible errors.

> In general $f(g(x)) \neq g(f(x))$. The order in which the functions are applied is important.

The $\neq$ symbol is read as 'is not equal to'.

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This is easily illustrated by the following example. When f is the function 'add 3' and g is the function 'square' then $g(f(x)) = (x + 3)^2$ but $f(g(x)) = x^2 + 3$. The only value of x for which $(x + 3)^2 = x^2 + 3$ is $x = 1$, as you can verify, so it is generally true that $f(g(x)) \neq g(f(x))$. There are *some* pairs of functions for which $f(g(x)) = g(f(x))$; for example if g is 'cube' and f is 'square' then $f(g(x)) = (x^3)^2 = x^6$ and $g(f(x)) = (x^2)^3 = x^6$, also. These are exceptions, however.

◆ Does $f(g(x)) = g(f(x))$ for the functions $g(x) = 2x + 1, f(x) = 3x + 2$?

◇ $f(g(x)) = f(u) = 3u + 2 = 3(2x + 1) + 2 = 6x + 3 + 2 = 6x + 5$

$g(f(x)) = g(u) = 2u + 1 = 2(3x + 2) + 1 = 6x + 4 + 1 = 6x + 5$

Hence $f(g(x)) = g(f(x))$ in this case. □

When dealing with a composite function $f(g(x))$, care must be taken to ensure that the function is only applied to admissible values of x, i.e. to values of x that are within the *domain* of the composite function.

Consider the following three examples.

1 $g(x) = x^2 + 1$, $f(x) = \log_e(x)$

Here $f(g(x)) = \log_e(x^2 + 1)$.

The logarithmic function $\log_e(x)$ is defined only if $x > 0$, that is, only if the input to the function is positive. However, in this case $x^2 + 1 > 0$ for all x, so the composite function is well defined for all (real) values of x.

2 $g(x) = x^2 - 1, f(x) = x^{1/2}$

Here $f(g(x)) = (x^2 - 1)^{1/2}$.

Any input value x will lead to a value of $g(x)$, but the input to the function f must be non-negative. Hence values of x that satisfy $x \geq 1$ or $x \leq -1$ are admissible, but values such that $-1 < x < 1$ are excluded from the domain of $f(g(x))$.

3 $g(x) = \sin(x), f(x) = \log_e(x)$

Here $f(g(x)) = \log_e(\sin(x))$.

The values of x for which $\sin(x) \leq 0$ are not admissible since they provide an unacceptable input for the $\log_e$ function.

The box diagram presents a useful check. In Figure 4 we show the function $\log_e(\sin(x))$.

The function $g(f(x)) = \sin(\log_e(x))$ presents no such problems.

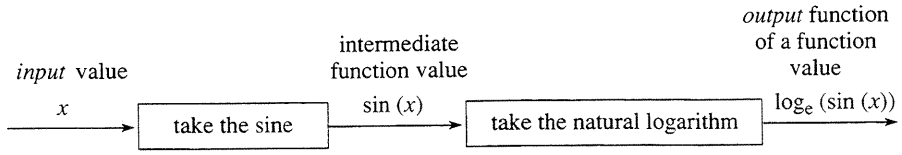


Figure 4 The function $\log_e(\sin(x))$.

◆ What are the entry requirements for each box in Figure 4?

◇ The input to the first box can be any real number but the input to the second box must be positive. Therefore, only those values of x which lead to a positive value of $\sin x$ are acceptable and will allow a successful evaluation of $\log_e(\sin(x))$. □

Question T3 If $f(x) = (x + 1)^{1/2}$ and $g(x) = \log_e x$ what are the restrictions on x in order that we may define $f(g(x))$ and $g(f(x))$? □

2.2 The chain rule

It is often necessary to differentiate a function of a function. This can sometimes be achieved by using basic differentiation techniques. Here, we will consider two special cases where this is possible, and then we will generalize the results to provide a powerful general rule (the *chain rule*) that can be used to differentiate any function of a function.

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1.1

Using the product rule

Consider the function $\sin^2 x$.

This can be regarded as a function of a function, for if we write $g(x) = \sin(x)$ and $f(x) = x^2$

then $f(g(x)) = (\sin x)^2 = \sin^2 x$

However, it can also be regarded as a product since $\sin^2 x = \sin x \times \sin x$, so it can be differentiated by means of the product rule to give

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x \times \sin x) = (\cos x \times \sin x) + (\sin x \times \cos x)$$

i.e. $$\frac{d}{dx}(\sin^2 x) = 2 \sin x \cos x$$

Now, the derivative of a function does not depend on how we choose to 'regard' the function. So, in this particular case, we have been able to use the product rule to find the derivative of a function of a function.

Similarly, the function $\cos^2 x$ can be considered as a product, so

$$\frac{d}{dx}(\cos^2 x) = \frac{d}{dx}(\cos x \times \cos x)$$

$$= (-\sin x \times \cos x) + [\cos x \times (-\sin x)] = 2 \cos x \times (-\sin x) = -2 \cos x \times \sin x$$

and $$\frac{d}{dx}(x^3 + 4)^2 = \frac{d}{dx}[(x^3 + 4) \times (x^3 + 4)]$$

$$= 3x^2 (x^3 + 4) + (x^3 + 4) \times 3x^2$$

$$= 2(x^3 + 4) \times 3x^2$$

In each of these three cases the pattern is the same; we start with an expression of the form

$$q(x) = (\text{function of } x)^2$$

and we find that

$$\frac{dq}{dx} = 2(\text{function of } x) \times (\text{derivative of function})$$

> So, if $q(x) = [p(x)]^2$
>
> then $q'(x) = 2p(x) \times p'(x)$

Question T4 Find the derivatives of the following composite functions:

(a) $q(x) = (2x - 3)^2$

(b) $q(x) = (3x^4 + 1)^2$

(c) $q(x) = (x + \sin x)^2$ □

Using the properties of the logarithmic function

The function $\log_e(x^3)$ can be regarded as a function of a function, but thanks to the special properties of logarithmic functions it can also be written as $3 \log_e x$, and it then follows from the constant multiple rule that

$$\frac{d}{dx}[\log_e(x^3)] = 3 \times \frac{d}{dx}[\log_e(x)] = 3 \times \frac{1}{x} = \frac{3}{x}$$

So, once again, we have been able to use a technique of basic differentiation to find the derivative of a function of a function. In this case it would have been tempting to have taken the result that the derivative of $\log_e(x)$ is $1/x$ and to have concluded that the derivative of $\log_e x^3$ is $1/x^3$. *But this is clearly wrong.* In fact, the correct answer, $\frac{3}{x} = \frac{1}{x^3} \times 3x^2$ and the 'correction factor' of $3x^2$ is the derivative of the function x^3.

◆ What is the derivative of $\log_e(x^5)$?

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◇ $\log_e(x^5) = 5\log_e x$ and the derivative is $5/x$. (Once again it is worth noticing that this answer may be written as $\frac{1}{x^5} \times (5x^4)$ where the final factor $(5x^4)$ is the derivative of the 'inner' function x^5.) □

chain rule
function of a function rule

We can summarize this result as follows:

if $q(x) = \log_e[p(x)]$

then $q'(x) = \frac{1}{p(x)} \times p'(x)$

The general case — the chain rule

The two boxed results given above are special cases of a general rule for differentiating functions of functions. That general rule is known as the *chain rule*.

The **chain rule**

If $q(x) = f(g(x))$

then $q'(x) = f'(g(x)) \times g'(x)$ (2a)

Alternatively, if we write

$u = g(x)$ and $q = f(u)$

then we can express the chain rule in the form

$$\frac{dq}{dx} = \frac{dq}{du} \times \frac{du}{dx} \quad (2b)$$

The chain rule, expressed in either of these equivalent forms is also known as the **function of a function rule**. You will also see the rule written in various 'mixed' notations such as $\frac{dq}{dx} = f'(u) \times \frac{du}{dx}$.

When we use the chain rule in the form of Equation 2b, it is almost as if we are multiplying together two fractions, dq/du and du/dx, and cancelling the du terms to leave dq/dx. We most definitely are *not* doing this, but it must be admitted that the idea makes the rule easier to remember!

All of the results obtained earlier in this subsection using the product rule, or the special properties of logarithmic functions, can also be obtained from the chain rule. For example, if we look again at the function $q(x) = \sin^2 x$ but we now identify $g(x) = u = \sin x$ and $q = u^2$, the chain rule tells us that

$$q'(x) = \frac{dq}{du} \times \frac{du}{dx} = 2u \times \cos x = 2 \sin x \cos x$$

Similarly, in the second example, if $q(x) = (x^3 + 4)^2$ and we identify $u = x^3 + 4$ and $q = u^2$

then $q'(x) = \frac{dq}{du} \times \frac{du}{dx} = 2u \times (3x^2) = 2(x^3 + 4) \times 3x^2 = 6x^2(x^3 + 4)$

Also, if $q(x) = \log_e(x^3)$ and we identify $u = x^3$ and $q = \log_e u$

then $q'(x) = \frac{dq}{du} \times \frac{du}{dx} = \frac{1}{u} \times (3x^2) = \frac{3x^2}{x^3} = \frac{3}{x}$

Moreover, the chain rule can be applied to many other composite functions that could not be differentiated using more elementary techniques. We can summarize the steps involved in applying the chain rule as follows:

To differentiate a function of a function $q(x) = f(g(x))$ using the chain rule:

1 *Identify* $u = g(x)$ and $q = f(u)$

2 *Differentiate* to find $\frac{du}{dx}$ and $\frac{dq}{du}$.

3 *Multiply* the last two expressions to find $\frac{dq}{dx}$.

4 *Rewrite* u in terms of x and simplify.

Pay particular attention to the last step in this procedure. Forgetting to re-express the final answer in terms of the original variables is a very common mistake, even among those who are proficient at differentiation.

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Here are some worked examples to show how this procedure works in practice.

Example 1 Find the derivative of $e^{\sin x}$.

Solution Let $u = \sin x$ and $q = e^u$

then $\dfrac{du}{dx} = \cos x, \quad \dfrac{dq}{du} = e^u$

so $\dfrac{dq}{dx} = \dfrac{dq}{du} \times \dfrac{du}{dx} = e^u \times \cos x$

thus $\dfrac{dq}{dx} = e^{\sin x} \times \cos x = \cos x\ e^{\sin x}$ □

Example 2 Find the derivative of $\sin(x^2 + 1)$.

Solution Let $u = x^2 + 1$ and $q = \sin u$

then $\dfrac{du}{dx} = 2x, \quad \dfrac{dq}{du} = \cos u$

so $\dfrac{dq}{dx} = \dfrac{dq}{du} \times \dfrac{du}{dx} = \cos u \times 2x$

thus $\dfrac{dq}{dx} = \cos(x^2 + 1) \times 2x = 2x \cos(x^2 + 1)$ □

Question T5 Using the chain rule, find the derivatives of the following functions:
(a) e^{2x}, (b) $\exp(x^2)$, (c) $(x^2 + 1)^5$, (d) $\sin(6x)$, (e) $\cos(3x)$, (f) $\log_e(e^x)$, (g) $\log_e(1 + x^2)$, (h) $\cos(\omega t + \phi)$. □

In practice you may have to deal with much nastier functions, and they may include functions of functions of functions or worse! This is where the chain rule really justifies its name since we can go on applying it as though adding links to a chain. For example, if $q = f(g(h(x)))$ then we can let $v = h(x)$, $u = g(v)$ and $q = f(u)$ and use the chain rule to obtain

$$\frac{dq}{dx} = \frac{dq}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = f'(u) \times g'(v) \times h'(x)$$

In practice, when faced with this kind of problem, it is often easier to just repeatedly apply the usual two-step form of the chain rule. Part (c) of the next question will give you the chance to try this for yourself.

◆ Differentiate the following composite functions, which all arise in physical problems, with respect to the indicated variable.

(a) The *Gaussian function*, $y = \dfrac{1}{\sigma\sqrt{2\pi}} \exp[-(x - m)^2/(2\sigma^2)]$, with respect to x, where σ, m and π are constants.

(b) The *Boltzmann factor*, $y = e^{-E/kT}$, with respect to T, where E and k are constants.

(c) The *Planck function*, $y = \dfrac{2hc^2}{\lambda^5} \dfrac{1}{(e^{hc/\lambda kT} - 1)}$, with respect to λ, where h, c, k and T are all constants.

◇ (a) $$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x - m)^2\ (2\sigma^2)] \times \frac{d}{dx}\left[\frac{-(x - m)^2}{2\sigma^2}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x - m)^2/(2\sigma^2)] \times \left[\frac{-2(x - m)}{2\sigma^2}\right]$$

$$= \frac{-(x - m)}{\sigma^3\sqrt{2\pi}} \exp[-(x - m)^2\ (2\sigma^2)]$$

(b) $$\frac{dy}{dT} = e^{-E/kT} \times \frac{d}{dT}\left(\frac{-E}{kT}\right) = \frac{E}{kT^2} e^{-E/kT}$$

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(c) Using the product rule, we obtain

$$\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{2hc^2}{\lambda^5}\right) \times \frac{1}{(e^{hc/\lambda kT} - 1)} + \frac{2hc^2}{\lambda^5} \times \frac{d}{d\lambda}\left[\frac{1}{(e^{hc/\lambda kT} - 1)}\right]$$

$$= \frac{-10hc^2}{\lambda^6} \times \frac{1}{(e^{hc/\lambda kT} - 1)} + \frac{2hc^2}{\lambda^5} \times \frac{d}{d\lambda}\left[(e^{hc/\lambda kT} - 1)^{-1}\right]$$

Now, applying the chain rule to the final term, we find

$$\frac{dy}{d\lambda} = \frac{-10hc^2}{\lambda^6} \times \frac{1}{(e^{hc/\lambda kT} - 1)}$$

$$+ \frac{2hc^2}{\lambda^5} \times \left[-(e^{hc/\lambda kT} - 1)^{-2}\right] \times \frac{d}{d\lambda}(e^{hc/\lambda kT} - 1)$$

Applying the chain rule again, to the new final term

$$\frac{dy}{d\lambda} = \frac{-10hc^2}{\lambda^6} \times \frac{1}{(e^{hc/\lambda kT} - 1)} + \frac{2hc^2}{\lambda^5} \times \left[\frac{-1}{(e^{hc/\lambda kT} - 1)^2}\right] \times e^{hc/\lambda kT} \frac{d}{d\lambda}\left(\frac{hc}{\lambda kT}\right)$$

$$= \frac{-10hc^2}{\lambda^6} \times \frac{1}{(e^{hc/\lambda kT} - 1)} + \frac{2hc^2}{\lambda^5} \times \left[\frac{-1}{(e^{hc/\lambda kT} - 1)^2}\right] \times e^{hc/\lambda kT}\left(\frac{-hc}{\lambda^2 kT}\right)$$

$$= \frac{2hc^2}{\lambda^6(e^{hc/\lambda kT} - 1)}\left[\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)} - 5\right] \quad \square$$

2.3 Differentiating implicit functions

implicit function
implicit differentiation

Functions are often specified *explicitly* in terms of equations or formulae such as $f(x) = ax^2$ or $\theta(t) = \omega t + \phi$, where the dependent variable is isolated on the left-hand side as the *subject* of the equation. However, it is also possible to specify a function *implicitly* by means of a formula that relates the dependent and independent variables, but does not isolate the dependent variable as the subject. For instance, the formula

$$\sqrt{\frac{y}{a}} - x = 0 \tag{3}$$

implicitly defines y as a function of x, even though it does not do so explicitly. In this particular case a little algebraic manipulation soon leads to the explicit relationship $y = ax^2$, but there are also cases where an implicitly defined function has no explicit representation at all. (☞) In general, a dependent variable y will be defined implicitly as a function of an independent variable x if x and y are related by an equation of the form

$$F(x, y) = 0$$

where $F(x, y)$ is a function of both x and y. A function $y(x)$ defined in this way is called an **implicit function**.

For example, the angular position of a planet moving around the Sun is determined from a quantity called the *eccentric anomaly* E that is defined implicitly as a function of time t by *Kepler's equation*

$$E - e\sin E = 2\pi(t - T)/a^{3/2}$$

where e, T and a are constants. Although it is possible to find approximate expressions for E, there is no exact explicit representation.

The chain rule makes it possible to differentiate implicit functions by means of a technique known as **implicit differentiation**. As a simple example, suppose that y is defined as an implicit function of x by the equation

$$y^3 = x^5$$

Differentiating both sides of this equation (with respect to x) gives us

$$\frac{d}{dx}(y^3) = 5x^4$$

But, if we put $u = y^3$ and use the chain rule $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$ we find

$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

thus $3y^2\frac{dy}{dx} = 5x^4$ and hence $\frac{dy}{dx} = \frac{5x^4}{3y^2}$

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We can now use the original relationship between x and y to express the right-hand side entirely in terms of x. In this case, $y^2 = x^{10/3}$

so $\dfrac{dy}{dx} = \dfrac{5}{3}x^{2/3}$

In physics, implicit differentiation often arises in proofs and derivations, and frequently involves functions that have no exact explicit representation.

Question T6 Use the technique of implicit differentiation to find dy/dx if $y^5 = \sin x$. Leave your answer as an expression in both x and y. □

2.4 Differentiating inverse functions

This subsection shows how the chain rule can be used to differentiate *inverse functions* such as the inverse trigonometric function $\arcsin x$. However, before introducing the general method for differentiating such functions we consider a simple problem that illustrates a useful ancillary rule.

◆ If $y = x^{1/2}$, write down dy/dx. Then write down an expression for x in terms of y, and hence find dx/dy as a function of y. Finally, express dx/dy in terms of x.

Remember that if $y = x^a$, then $dy/dx = ax^{(a-1)}$.

◇ Using the rule for differentiating powers, $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

$x = y^2$, so $\dfrac{dx}{dy} = 2y$. In terms of x, $\dfrac{dx}{dy} = 2\sqrt{x}$. □

Notice that the expression for $\dfrac{dy}{dx}$ in the last question is the reciprocal of that for $\dfrac{dx}{dy}$. This is illustrative of a very useful general rule, the **inversion rule**:

inversion rule

the *inversion rule*: $$\frac{dx}{dy} = 1\Big/\left(\frac{dy}{dx}\right) \qquad (4)$$

Check that this rule works by trying some other functions that involve powers of x. A general proof is given in the module devoted to *solving first-order differential equations*. See *inversion rule* in the *Glossary* for details.

This looks as if we are finding the reciprocal of a fraction dy/dx. That is definitely *not* what we are doing, but it does make the rule easy to remember.

In a few cases the inversion rule is sufficiently powerful on its own to enable us to differentiate some inverse functions. For instance, using the inversion rule we can differentiate $\log_e x$, the inverse function of e^x. If $y = \log_e x$, then $x = e^y$ and $dx/dy = e^y$. Now, using this result together with the inversion rule,

$$\frac{d(\log_e x)}{dx} = \frac{dy}{dx} = 1\Big/\left(\frac{dx}{dy}\right) = \frac{1}{e^y} = \frac{1}{x}$$

Often, though, the inversion rule is not enough and we have to use the chain rule to differentiate inverse functions. As an example, consider the problem of finding the derivative dy/dx when

$$y = \arcsin x$$

The graph of this function is shown in Figure 5. Taking the sine of both sides gives us

$$\sin y = \sin(\arcsin x) = x$$

Now differentiate both the right and left-hand sides of this equation with respect to x. The right-hand side is simply 1, but to differentiate the left-hand side we have to apply the chain rule.

If we let $z = \sin y$, $\dfrac{dz}{dx} = \dfrac{dz}{dy} \times \dfrac{dy}{dx} = \cos y \times \dfrac{dy}{dx}$

Hence, the result of differentiating both sides of the equation $\sin y = x$ is

$$\cos y \times \frac{dy}{dx} = 1$$

so that $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$

The inverse trigonometric function $\arcsin x$ is defined for $-1 \le x \le 1$ by the requirement that if $y = \arcsin x$, then y is that value in the range $-\pi/2$ to $\pi/2$ such that $\sin y = x$.

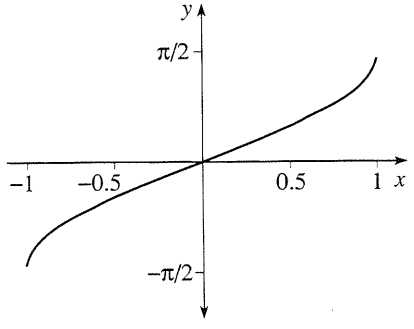


Figure 5 The graph of $y = \arcsin x$.

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All that remains is to express the right-hand side in terms of x. To do this note that $\cos^2 y = 1 - \sin^2 y = 1 - x^2$, so $\cos y = \pm\sqrt{1-x^2}$.

Now, an examination of Figure 5 shows that the graph of $y = \arcsin x$ has a *positive* gradient throughout its domain of definition, so only *positive* values of $\cos y$ are admissible and hence we must use the positive square root when writing the final answer.

Thus $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Table 2 records this result together with the derivatives of some other inverse functions.

Finally let us look at a more complicated example that uses both the chain rule and the inversion rule.

Example 3 Figure 6 shows the trajectory of a falling object released at point O from an aircraft moving horizontally with a constant speed u. The tangent at any point on the trajectory is inclined at an angle θ to the horizontal as shown, and this angle increases throughout the fall. What is the rate of change of θ with time?

Solution Assuming that air resistance may be neglected, and using the coordinate system shown in Figure 6, in which x increases to the right and y increases downwards, the components of velocity in the horizontal and vertical direction will be

$$\frac{dx}{dt} = u \quad \text{and} \quad \frac{dy}{dt} = gt$$

where g is the magnitude of the acceleration due to gravity.

Now, you should be able to see from Figure 6 that $\tan\theta = dy/dx$

But using the chain rule and the inversion rule we find

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{gt}{u}$$

Thus, $$\tan\theta = \frac{gt}{u}$$

and from this we obtain $$\theta = \arctan\left(\frac{gt}{u}\right)$$

So, to find the rate of change of θ with t, i.e. $\dfrac{d\theta}{dt}$, we need to differentiate the inverse function $\arctan\left(\dfrac{gt}{u}\right)$.

To do this we first take the tangent of both sides:

$$\tan\theta = \tan\left[\arctan\left(\frac{gt}{u}\right)\right] = \frac{gt}{u}$$

and then differentiate with respect to t. The right-hand side gives g/u, so applying the chain rule to the left-hand side we obtain

$$\sec^2\theta \times \frac{d\theta}{dt} = \frac{g}{u}$$

so $$\frac{d\theta}{dt} = \frac{g}{u\sec^2\theta}$$

Now we only need to express $\sec^2\theta$ in terms of t, and we can do this using a standard trigonometric identity and the expression for $\tan\theta$ we obtained earlier

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{g^2t^2}{u^2}$$

Thus $$\frac{d\theta}{dt} = \frac{g}{u} \times \frac{1}{\left(1 + \dfrac{g^2t^2}{u^2}\right)} = \frac{gu}{u^2 + g^2t^2} \quad \square$$

Table 2 Derivatives of some inverse functions.

$f(x)$	$f'(x)$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\operatorname{arcsec} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\operatorname{arccosec} x$	$\dfrac{-1}{x\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$\dfrac{-1}{1+x^2}$

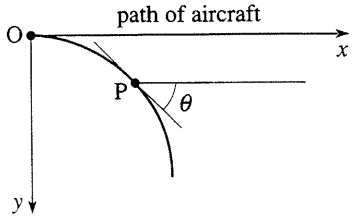


Figure 6 Motion of an object released from an aircraft.

Trigonometric identities are fully discussed elsewhere in *FLAP*. See the *Glossary* for details.

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A word of caution If you differentiate the function $\theta = \arctan(t)$ then, as Table 2 indicates, you should find that $\dfrac{d\theta}{dt} = \dfrac{1}{1+t^2}$. It would be quite wrong to conclude from this that the process of differentiating $\theta = \arctan\left(\dfrac{gt}{u}\right)$ can be accomplished by simply replacing t on the right-hand side by gt/u. You can obtain the right answer by replacing t by gt/u on both sides, but the safest way is to go through the full method as we have just done. □

parameter
parametric equations
parametric functions
parametric differentiation

Question T7 Find the derivative of each of the following functions:
(a) $\arcsin(x^2)$, (b) $\arccos(x^2+4)$ (☞), (c) $\arctan(3-x)$ □

You may find it helpful to know that the gradient of $\arccos x$ is negative throughout its domain of definition. You will also need to think carefully about part (b).

2.5 Differentiating parametric functions

It is sometimes useful to express two variables x and y each in terms of a third variable called a **parameter**. For example, in *coordinate geometry* the equation that represents a circle of radius a, centred on the origin, is usually written in the form

$$x^2 + y^2 = a^2$$

However, thanks to the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, the same circle can also be represented in terms of the parameter θ by the pair of equations

$$x = a\cos\theta \tag{5a}$$

$$y = a\sin\theta \tag{5b}$$

where θ may take any value in the range 0 to 2π.

Coordinate geometry is discussed elsewhere in *FLAP*. See the *Glossary* for details.

Eliminating θ from Equations 5a and 5b (by squaring and adding the two equations) immediately yields the usual equation of the circle. Equations 5a and 5b are called the **parametric equations** of the circle. Functions defined in terms of parametric equations are called **parametric functions**. Whenever we describe the motion of a moving object by specifying its position coordinates x, y and z as functions of time, we are really writing down the parametric equations of the object's pathway, with time t as the parameter.

Returning to the case of the circle $x^2 + y^2 = a^2$, if we want to find dy/dx, the gradient of the tangent at a point on the circle, we can use the parametric equations and employ a variation of the chain rule, namely

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} \tag{6}$$

Now, $\dfrac{dy}{d\theta} = a\cos\theta$ and $\dfrac{dx}{d\theta} = -a\sin\theta$

so we obtain $\dfrac{dy}{dx} = \dfrac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

This is an example of **parametric differentiation**. (☞)

An equivalent result $\dfrac{dy}{dx} = -\dfrac{x}{y}$ can be obtained by applying the technique of *implicit differentiation* to the usual equation of a circle.

Example 4 As illustrated in Figure 7, a projectile is fired into the air at an angle of 60° to the horizontal with a speed of $u = 6\ \mathrm{m\,s^{-1}}$, so its initial horizontal velocity is $u_x = u\cos(60°) = 3\ \mathrm{m\,s^{-1}}$, and its initial vertical velocity is $u_y = u\sin(60°) = 3\sqrt{3}\ \mathrm{m\,s^{-1}}$. What is the flight time for the projectile to reach the top of its trajectory (i.e. how long does it take to reach the point at which the tangent to the trajectory is horizontal)?

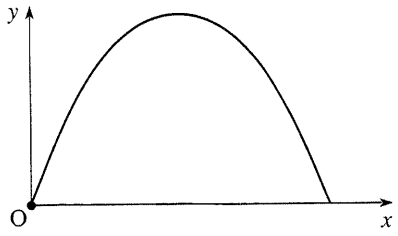


Figure 7 Trajectory of a projectile.

Solution If we neglect air resistance, and take upwards as the positive y-direction, the equations describing its trajectory are (☞)

$$x = u_x t \quad \text{and} \quad y = u_y t - \tfrac{1}{2}gt^2$$

If you are unfamiliar with the physics of projectile motion, just take these first two equations on trust.

where x and y are, respectively, the horizontal and vertical displacements from the point of projection at time t, and g is the magnitude of the acceleration due to gravity. The horizontal and vertical components of the velocity are given by

$$\frac{dx}{dt} = u_x \quad \text{and} \quad \frac{dy}{dt} = u_y - gt$$

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At the top of the trajectory the tangent to the trajectory is horizontal, therefore $\frac{dy}{dx} = 0$ but, using the chain rule and the inversion rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{u_y - gt}{u_x}$$

so at the top of the trajectory

$$\frac{u_y - gt}{u_x} = 0, \text{ i.e. } t = \frac{u_y}{g} \quad \square$$

Question T8 Find the derivative dy/dx for each of the following pairs of parametric equations:

(a) $x = a\cos\theta, y = b\sin\theta$

(b) $x = at^2, y = 2at$

(c) $x = ct, y = c/t$ $\square$

second derivative
differential operator

3 Second and higher derivatives

3.1 The derivative of a derivative

In the same way that we can differentiate a function $f(x)$ to obtain its derivative (or derived function) $f'(x)$, so we can differentiate the function $f'(x)$ to obtain its derivative which is denoted $f''(x)$. This is called the **second derivative** of $f(x)$.

$f''(x)$ should be read as 'f double prime of x'.

Hence if $f(x) = 4x^3 + 7x^2 + 6x - 8$ it follows that

$$f'(x) = 12x^2 + 14x + 6$$

and $$f''(x) = 24x + 14$$

If we write $y = f(x)$ and denote its derivative (sometimes called its *first derivative* in this context) by dy/dx, then an alternative notation for the second derivative is $\frac{d^2y}{dx^2}$.

$\frac{d^2y}{dx^2}$ is read as 'dee two y by dee x squared'.

For example, if $y = 4\sin(2x) + e^{3x}$, then the first derivative is

$$\frac{dy}{dx} = 8\cos(2x) + 3e^{3x}$$

and the second derivative is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 8 \times [-2\sin(2x)] + 3 \times (3e^{3x})$$

i.e. $$\frac{d^2y}{dx^2} = -16\sin(2x) + 9e^{3x}$$

Notice that the notation $\frac{d^2y}{dx^2}$ is a rather natural one since it suggests that the operation of differentiation has been repeated twice. Indeed, the symbol d/dx is sometimes referred to as a **differential operator** since it conveys the instruction to differentiate whatever appears immediately to its right. Also notice the precise position of the superscript 2s in the second derivative symbol; it is the differential operator that is 'squared', not the independent variable y.

Some important physical quantities are often thought of as second derivatives. For instance, in *kinematics* — the study of motion — *velocity* is defined as the rate of change of *position*, so $v_x = \frac{dx}{dt}$ for example, but *acceleration* is defined as the rate of change of velocity, so $a_x = \frac{dv_x}{dt}$ and consequently we can regard acceleration as a second derivative, with $a_x = \frac{d^2x}{dt^2}$. Indeed, derivatives with respect to time arise so often in physics that you will often see them denoted by a special notation that uses a dot to indicate differentiation; thus $v_x = \dot{x}(t)$ and $a_x = \ddot{x}(t)$.

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Question T9 Find $f''(x)$ for each of the following functions:

(a) $f(x) = x^6 - 2x^2 + 7$

(b) $f(x) = ax + b$, where a and b constants

(c) $f(x) = a\sin x + b\cos x$

(d) $f(x) = e^{3x}$

(e) $f(x) = \log_e(x^2 + 1)$

(f) Rewrite your answers to parts (a) to (e) using $\frac{d^2y}{dx^2}$ notation. □

Question T10 (a) If the x-coordinate of a moving object is given as a function of time by $x(t) = pt + qt^2$, find expressions for the components of velocity and acceleration in the x-direction.

(b) Find $\ddot{x}(t)$ when $x(t) = A\sin(\omega t)$ and when $x(t) = B\cos(\omega t)$. □

3.2 Higher derivatives

higher derivative

Assuming that the necessary derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. The third derivative is the derivative of the second derivative, the fourth derivative is the derivative of the third derivative and so on. All these are known collectively as **higher derivatives**, and we define them by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$, $\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$, and so on.

In general, the n^{th} derivative is defined by $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$ (7)

The n^{th} derivative is sometimes called the n^{th} *order* derivative.

To be consistent with this notation we should write $\frac{dy}{dx}$ as $\frac{d^1y}{dx^1}$ but this is never done in practice.

In the alternative functional notation, derivatives higher than the third are usually denoted by lower case Roman numerals: $f'(x), f''(x), f'''(x), f^{(iv)}(x)$ ..., but the n^{th} derivative is usually denoted $f^{(n)}(x)$.

Occasionally you may see $f(x)$ written as $f^{(0)}(x)$, usually to obtain a neat form for a general formula. For example, in the series

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

when $n = 0$.

Example 5 Find the first three derivatives of $y = \frac{1}{x^3}$.

Solution $\frac{dy}{dx} = \frac{-3}{x^4}$, $\frac{d^2y}{dx^2} = \frac{+12}{x^5}$, $\frac{d^3y}{dx^3} = \frac{-60}{x^6}$ □

◆ Find the first three derivatives of x^n, where $n \geq 3$. What happens if $n = 2$?

◇ $\frac{dy}{dx} = nx^{n-1}$, $\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$, $\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$

If $n = 2$ the derivatives are $2x$, 2, and 0. □

Example 6 Find the first four derivatives of $f(x) = 3x^4 - 8x^3 + 5x^2 - 6$.

Solution $f'(x) = 12x^3 - 24x^2 + 10x$

$f''(x) = 36x^2 - 48x + 10$

$f'''(x) = 72x - 48$

$f^{(iv)}(x) = 72$ (☞) □

Note that $f^{(n)}(x) = 0$ for $n \geq 5$.

Example 7 Find the n^{th} derivative of $f(x) = e^{3x}$.

Solution The first, second and third derivatives are

$f'(x) = 3e^{3x}$, $f''(x) = 3 \times 3e^{3x} = 3^2e^{3x}$, and $f'''(x) = 3^2 \times 3e^{3x} = 3^3e^{3x}$.

It is clear from these that in general, the n^{th} derivative is given by

$f^{(n)}(x) = 3^ne^{3x}$ □

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Question T11 Find the first four derivatives and the n^{th} derivative of the following functions:

(a) $y = f(x) = 4e^{5x}$

(b) $y = 3x^3$

(c) $y = A\cos(2x) + B\sin(2x)$ where A and B are constants.

(d) $y = \log_e x$ where $x \neq 0$ □

3.3 Second derivatives of implicit and parametric functions

The techniques we have accumulated so far to obtain $f'(x)$ from $f(x)$ can, in most cases, be applied to $f'(x)$ to obtain $f''(x)$ with no major difficulties. There are, however, two points which need great care. First is the fact that

$\frac{dx}{dy} = 1\Big/\left(\frac{dy}{dx}\right)$ is an exception

$\frac{d^2x}{dy^2}$ and $1\Big/\left(\frac{d^2y}{dx^2}\right)$ are *not* generally equal.

Consider, for example the relationship

$$y = \frac{1}{x}$$

Differentiating this expression twice we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = +\frac{2}{x^3}$$

Now, if $x = \frac{1}{y}$, $\frac{dx}{dy} = -\frac{1}{y^2}$ and $\frac{d^2x}{dy^2} = +\frac{2}{y^3}$

So, while it is true that $\frac{dx}{dy} = \frac{-1}{y^2} = -x^2 = 1\Big/\left(\frac{dy}{dx}\right)$

it is also the case that $\frac{d^2x}{dy^2} \neq 1\Big/\left(\frac{d^2y}{dx^2}\right)$ (☞)

Note that we have used the symbol ≠ to show that the two terms are unequal.

since $\frac{d^2x}{dy^2} = \frac{2}{y^3} = 2x^3$, whereas $1\Big/\left(\frac{d^2y}{dx^2}\right) = 1\Big/\left(\frac{2}{x^3}\right) = \frac{x^3}{2}$

We also need to take great care over parametric differentiation. We have seen that for the pair of equations

$$x = a\cos\theta \quad \text{and} \quad y = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

But to obtain $\frac{d^2y}{dx^2}$ we need to determine $\frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\theta)$.

This can be done with the aid of the chain rule, either by writing $z = -\cot\theta$ and using

$$\frac{dz}{dx} = \frac{dz}{d\theta} \times \frac{d\theta}{dx}$$

or more directly, by writing

$$\frac{d}{dx}(-\cot\theta) = \frac{d}{d\theta}(-\cot\theta) \times \frac{d\theta}{dx}$$

In either case we use the inversion rule to replace $d\theta/dx$ by the reciprocal of $dx/d\theta$ and hence obtain

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\cot\theta)\Big/\frac{dx}{d\theta} = \frac{+\mathrm{cosec}^2\,\theta}{-a\sin\theta} = -\frac{1}{a}\mathrm{cosec}^3\,\theta$$

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Example 8 If $x = at^2$ and $y = 2at$, determine $\dfrac{d^2y}{dx^2}$.

Solution $\dfrac{dx}{dt} = 2at$ and $\dfrac{dy}{dt} = 2a$

so $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

hence $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$$

using the chain rule, we find

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{dt}{dx} = -\frac{1}{t^2} \times \frac{dt}{dx}$$

and using the inversion rule, we find

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2} \Big/ \frac{dx}{dt} = \frac{-1}{t^2}\frac{1}{2at} = \frac{-1}{2at^3}$$

It is worth observing that implicit differentiation provides an alternative way of dealing with this example. To adopt this approach start by noting that $y^2 = 4a^2t^2 = 4a \times at^2 = 4ax$.

So implicit differentiation of this with respect to x gives us

$$2y\frac{dy}{dx} = 4a$$

A second implicit differentiation, in which $y\dfrac{dy}{dx}$ is treated as a product, gives

$$2y\frac{d^2y}{dx^2} + \frac{2dy}{dx} \times \frac{dy}{dx} = 0$$

so $$2 \times 2at \times \frac{d^2y}{dx^2} + 2\frac{1}{t} \times \frac{1}{t} = 0$$

i.e. $$4at\frac{d^2y}{dx^2} + \frac{2}{t^2} = 0$$

hence, as before $$\frac{d^2y}{dx^2} = \frac{-2}{t^2} \times \frac{1}{4at} = \frac{-1}{2at^3}$$ □

Question T12 Find $\dfrac{d^2y}{dx^2}$ for each of the following pairs of parametric equations:

(a) $x = \cos\theta$, $y = \sin\theta$

(b) $x = a\cos\theta$, $y = b\sin\theta$

(c) $x = ct$, $y = c/t$ □

4 A further example

We will end by tackling a set of related electrical problems — all of which involve a circuit containing an inductor and capacitor (an *LC* circuit). Working through each of the steps in these problems for yourself will help you to review what you have learned in this module.

4.1 An example from electrical circuits

In a *series LC circuit* the charge $q(t)$ on the capacitor at time $t > 0$ satisfies the equation

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad (8)$$

where L and C are constants, known as the inductance and capacitance, respectively.

You do not need to be familiar with the physics of electrical circuits to follow the essential steps of this discussion.

The first problem we consider is that of showing that

$$q(t) = A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right) \qquad (9)$$

satisfies Equation 8 for arbitrary values of the constants A and B.

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To do this we need to deduce an expression for d^2q/dt^2 from Equation 9 and substitute it into Equation 8. The first step is to find dq/dt from Equation 9.

Now, $q(t)$ is the sum of two terms, each of which is a function of a function. Taking just the second term and calling it y we have $y = B\sin\left(\frac{t}{\sqrt{LC}}\right)$ and if we put $u = \frac{t}{\sqrt{LC}}$ we find $y = B\sin u$. From this, we see $\frac{du}{dt} = \frac{1}{\sqrt{LC}}$ and $\frac{dy}{du} = B\cos u$, and if we then use the chain rule, we find

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = B\cos u \times \frac{1}{\sqrt{LC}} = \frac{B}{\sqrt{LC}}\cos u = \frac{B}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

◆ If we now repeat the process for the first term in Equation 9, what is the derivative of the term

$$y = A\cos\left(\frac{t}{\sqrt{LC}}\right)$$

◇ If we put $y = A\cos u$, where $u = \frac{t}{\sqrt{LC}}$, it follows from the chain rule that

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = -A\sin u \times \frac{1}{\sqrt{LC}} = -\frac{A}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \quad \square$$

◆ From the above discussion, what is dq/dt from Equation 9?

◇ $\frac{dq}{dt} = \frac{-A}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{B}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$. $\square$

So we now know dq/dt. We now need to find $\frac{d^2q}{dt^2}$

◆ What is the derivative of $y = \frac{-A}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$?

◇ Putting $y = \frac{-A}{\sqrt{LC}}\sin u$, where $u = \frac{t}{\sqrt{LC}}$, we find using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = \frac{-A}{\sqrt{LC}}\cos u \times \frac{1}{\sqrt{LC}} = \frac{-A}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) \quad \square$$

◆ What is the derivative of $y = \frac{B}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$?

◇ Using the same method we find, $\frac{dy}{dt} = \frac{-B}{LC}\sin\left(\frac{t}{\sqrt{LC}}\right)$. $\square$

◆ What is $\frac{d^2q}{dt^2}$?

◇ $\frac{d^2q}{dt^2} = \frac{-A}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{B}{LC}\sin\left(\frac{t}{\sqrt{LC}}\right)$. $\square$

Therefore, if we substitute this result (and Equation 9) into Equation 8, we find

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = L\left[\frac{-A}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{B}{LC}\sin\left(\frac{t}{\sqrt{LC}}\right)\right] + \frac{1}{C}\left[A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right)\right]$$

$$= \frac{-A}{C}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{B}{C}\sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{A}{C}\cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{B}{C}\sin\left(\frac{t}{\sqrt{LC}}\right) = 0$$

Thus the given expression (Equation 9) does satisfy the equation $L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$, irrespective of the values of A and B.

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Now we can consider a related problem. Given that Equation 9 provides a solution to Equation 8 for *any* values of A and B, what are the specific choices of A and B needed to ensure that at time $t = 0$ the charge on the capacitor has the value q_0 and the rate of flow of charge in the circuit is zero? In other words, what choices of A and B ensure that

$$q(0) = q_0 \quad \text{and} \quad \frac{dq}{dt}(0) = 0$$

To satisfy the second of these conditions we require

$$-\frac{A}{\sqrt{LC}}\sin(0) + \frac{B}{\sqrt{LC}}\cos(0) = 0$$

i.e. $$\frac{B}{\sqrt{LC}} = 0$$

We must therefore choose $B = 0$, so $q = A\cos\left(\frac{t}{\sqrt{LC}}\right)$

To satisfy the additional condition that $q(0) = q_0$ we require

$$A\cos(0) = q_0$$

i.e. $A = q_0$

so the solution satisfying both the given initial conditions is

$$q(t) = q_0\cos\left(\frac{t}{\sqrt{LC}}\right)$$

5 Closing items

5.1 Module summary

1 A *function of a function* produces an output by applying *first* one process, *then* another to an input.

2 In general $f(g(x)) \neq g(f(x))$. The order in which the functions are applied is important.

3 When dealing with a *composite function* $f(g(x))$, care must be taken to ensure that the function is only applied to admissible values of x, i.e. to values of x that are within the *domain* of the composite function.

4 A function of a function can be differentiated using the *chain rule*, which states that if $u = g(x)$ and $q = f(u)$, then

$$\frac{dq}{dx} = \frac{dq}{du} \times \frac{du}{dx} = f'(u) \times g'(x) \qquad \text{(Eqn 2)}$$

5 An *implicit function* is a function that is defined by means of a relation such as $F(x, y) = 0$. The chain rule makes it possible to differentiate implicit functions by the technique of *implicit differentiation*.

e.g. if $y^3 = x^5$ then $3y^2\frac{dy}{dx} = 5x^4$, so $\frac{dy}{dx} = \frac{5x^4}{3y^2} = \frac{5}{3}x^{2/3}$

6 According to the *inversion rule*

$$\frac{dx}{dy} = 1\Big/\left(\frac{dy}{dx}\right) \qquad \text{(Eqn 4)}$$

7 *Inverse functions* can be differentiated by applying the inversion rule and the chain rule.

8 If x and y are both defined in terms of a *parameter* θ, then the derivative $\frac{dy}{dx}$ may be determined by the technique of *parametric differentiation*

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} \qquad \text{(Eqn 6)}$$

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9 If $y = f(x)$, then the n^{th} *derivative* of y is defined by

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \qquad \text{(Eqn 7)}$$

The first four derivatives of $f(x)$ may also be represented by $f'(x), f''(x), f'''(x)$, $f^{(\text{iv})}(x)$, and the n^{th} derivative by $f^{(n)}(x)$.

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Identify a function of a function.

A3 Use the chain rule to find the derivative of a function of a function.

A4 Find the derivative of an inverse function.

A5 Find the derivative of a function defined parametrically or implicitly.

A6 Find the second derivative of a function of a function.

A7 Find the second derivative of an inverse function.

A8 Find the second derivative of a function defined parametrically or implicitly.

A9 Find the n^{th} derivative of a function.

A10 Recognize and use the different notations for higher derivatives.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 *(A2)* Which of the following is a function of a function? If it is necessary to restrict the values of x, then say so.

(a) $F(x) = \sin(3x)$

(b) $F(x) = \log_e(1 - x^2)$

(c) $y^3 = x$.

Question E2 *(A2 and A3)* Find the derivative of:

(i) each of the functions in Question E1,

(ii) $f(g(x))$ and $g(f(x))$, where $u = g(x) = x^3 + 2$ and $f(x) = 5x + 4$.

Question E3 *(A2, A3 and A4)* Find the derivative dy/dx of $y = \arccos(3x - 2)$.

Question E4 *(A2, A3 and A5)* Find the derivative dy/dx when

$$x = 2 + 3\sqrt{2}t \quad \text{and} \quad y = 1 + 3\sqrt{2}t - 4.9t^2$$

Questions E5 *(A2 to A8)* Find the second derivatives of the functions in Questions E1, E3 and E4. If it is necessary to restrict the values of x, then say so.

Question E6 *(A6 and A9)* Find the first five derivatives and the n^{th} derivative of the following functions:

(a) $f(x) = x^6 - 2x^4 - x^3 + 6x$

(b) $f(x) = xe^x$

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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1.1

6 Answers and comments

Fast track answers

F1 (a) $F'(x) = 3\cos x - 4\sin x$

$F''(x) = -3\sin x - 4\cos x$

$F'''(x) = -3\cos x + 4\sin x$

$F^{(iv)}(x) = 3\sin x + 4\cos x$

(b) $\dfrac{dy}{dx} = \dfrac{1}{x}, \quad \dfrac{d^2y}{dx^2} = \dfrac{-1}{x^2}, \quad \dfrac{d^3y}{dx^3} = \dfrac{2}{x^3}, \quad \dfrac{d^4y}{dx^4} = \dfrac{-6}{x^4}$

Comment The two notations used in parts (a) and (b) are equivalent, i.e. $F'(x)$ is equivalent to dy/dx.

F2 $\dfrac{dx}{d\theta} = a(1-\cos\theta)$ and $\dfrac{dy}{d\theta} = a\sin\theta$

so $\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\times\dfrac{d\theta}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \dfrac{\sin\theta}{1-\cos\theta}$

i.e. $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1-\cos\theta}$

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{\sin\theta}{1-\cos\theta}\right) = \dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{1-\cos\theta}\right)\times\dfrac{d\theta}{dx}$

so $\dfrac{d^2y}{dx^2} = \left[\dfrac{(1-\cos\theta)\cos\theta - \sin\theta(+\sin\theta)}{(1-\cos\theta)^2}\right]\times\dfrac{d\theta}{dx}$

i.e. $\dfrac{d^2y}{dx^2} = \left[\dfrac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}\right]\Big/\dfrac{dx}{d\theta}$

$$\frac{d^2y}{dx^2} = \left[\frac{\cos\theta - 1}{(1-\cos\theta)^2}\right]\times\frac{1}{a(1-\cos\theta)} = \frac{-1}{a(1-\cos\theta)^2}$$

F3 (a) Put $u = \alpha t + \sin(\omega t)$ and $F(u) = e^u$

then $\dfrac{du}{dt} = \alpha + \omega\cos(\omega t)$ and $F'(u) = e^u$

Hence, by the *chain rule*

$$\frac{dF}{dt} = \frac{dF}{du}\times\frac{du}{dt} = e^u\times[\alpha + \omega\cos(\omega t)]$$
$$= [\alpha + \omega\cos(\omega t)]e^u$$
$$= [\alpha + \omega\cos(\omega t)]\times\exp[\alpha t + \sin(\omega t)]$$

i.e. $\dfrac{dF}{dt} = [\alpha + \omega\cos(\omega t)]\times\exp[\alpha t + \omega\sin(\omega t)]$

By the *product rule*

$$\frac{d^2F}{dt^2} = [\alpha + \omega\cos(\omega t)]\times\frac{d}{dt}[\exp[\alpha t + \omega\sin(\omega t)]]$$
$$+ [-\omega^2\sin(\omega t)]\times\exp[\alpha t + \omega\sin(\omega t)]$$

i.e. $\dfrac{d^2F}{dt^2} = [\alpha + \omega\cos(\omega t)]\times[\alpha + \omega^2\cos(\omega t)]$
$\times\exp[\alpha t + \omega\sin(\omega t)]$
$- \omega^2\sin(\omega t)\times\exp[\alpha t + \omega\sin(\omega t)]$

so $\dfrac{d^2F}{dt^2} = \{[\alpha + \omega\cos(\omega t)]\times[\alpha + \omega^2\cos(\omega t)]$
$- \omega^2\sin(\omega t)]\}\times\exp[\alpha + \omega\sin(\omega t)]$

(b) Put $u = \sin^2 x + 1$ and $F(u) = 5\log_e u$

then, using either the product rule ($\sin^2 x = \sin x\times\sin x$) or the chain rule, we find

$$\frac{du}{dx} = 2\sin x\cos x \quad \text{and} \quad F'(u) = \frac{5}{u}$$
$$\frac{dF}{dx} = \frac{dF}{du}\times\frac{du}{dx} = \frac{5}{u}\times(2\sin x\cos x)$$
$$= \frac{5\times 2\sin x\cos x}{\sin^2 x + 1} = \frac{10\sin x\cos x}{\sin^2 x + 1}$$

Applying the quotient rule we find

$$\frac{d^2F}{dx^2} = [(\sin^2 x + 1)\times\frac{d}{dx}(10\sin x\cos x)$$
$$- 10\sin x\cos x(2\sin x\cos x)] / (\sin^2 x + 1)^2$$
$$= \{(\sin^2 x + 1)[10\cos x\times\cos x + 10\sin x\times(-\sin x)]$$
$$- 10\sin x\cos x(2\sin x\cos x)\} / (\sin^2 x + 1)^2$$
$$= \frac{10(\sin^2 x + 1)(\cos^2 x - \sin^2 x) - 20\sin^2 x\cos^2 x}{(\sin^2 x + 1)^2}$$

(c) Let $y = \arcsin(5x)$ then $\sin y = 5x$.

Hence, differentiating implicitly with respect to x, we find

$$\cos y\times\frac{dy}{dx} = 5$$

so $\dfrac{dy}{dx} = \dfrac{5}{\cos y} = \dfrac{5}{\sqrt{1-25x^2}}$

Now put

$$\frac{dy}{dx} = \frac{5}{\sqrt{1-25x^2}} = 5(1-25x^2)^{-1/2} = G(x)$$

let $u = 1 - 25x^2$ and $g(u) = 5u^{-1/2}$

so $\dfrac{du}{dx} = -50x$ and $\dfrac{dg}{du} = -\dfrac{1}{2}\times 5u^{-3/2} = \dfrac{-5}{2u^{3/2}}$

Hence, by the chain rule

$$\frac{d^2F}{dx^2} = \frac{dG}{dx} = \frac{dg}{du}\times\frac{du}{dx}$$
$$= \frac{-5}{2u^{3/2}}\times(-50x) = \frac{125x}{(1-25x^2)^{3/2}}$$

Ready to study answers

R1 (a) $f(x)$ is an acceptable function of x and there are no restrictions on the value of x. Each value of x corresponds to a single value of y.

(b) y is an acceptable function of x, provided x is in the range $-1 \le x \le 1$ (this defines the *domain* of the function). There are many different values of y that could correspond to a given value of x, so to ensure that each admissible value of x corresponds to only a single value of y it is also necessary to impose restrictions on y in this case. Conventionally, y is required to be in the range $-\pi/2 \le y \le +\pi/2$, i.e. $-90° \le y \le +90°$ (this defines the *codomain* of the function).

(c) y is not a function of x since the given formula implies that $y = \pm 2x$, so it is *not* the case that a single value of y corresponds to each value of x.

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R2 (a) $x = (y/9)^{1/3}$ (b) $x = \arctan y$

(c) $2x = \log_e y$ so, $x = (\log_e y)/2$

(d) $3x = e^y$, so $x = e^y/3$

(If you are unsure of the answers to Questions R1 or R2, consult the entries on *functions*, *trigonometric functions*, *exponential functions*, *logarithmic functions* and *inverse functions* in the *Glossary*.)

R3 If the expression $\dfrac{f(a+h)-f(a)}{h}$ approaches a unique limiting value as h approaches zero, then that limiting value is called the *derivative* of $f(x)$ at $x = a$ and is written $f'(a)$.

R4 (a) $f'(x) = 48x^5 + 18x^2 - 10x$
(*differentiation* of *powers*).

(b) $f'(x) = 3\cos x - 4\sin x$
(*differentiation* of *trigonometric functions*)

R5 (a) $\dfrac{dy}{dx} = 6e^x$ (*differentiation* of *exponential functions*)

(b) $\dfrac{dy}{dx} = 2x - \dfrac{5}{x}$ (*differentiation* of *logarithmic functions*)

R6 (a) $x^2 \times e^x + 2xe^x = (x^2 + 2x)e^x$ (*product rule*)

(b) $x^3\cos x + 3x^2\sin x$ (*product rule*)

(c) $e^x \times (-\sin x) + (e^x) \times \cos x$ (*product rule*)
$= e^x(-\sin x + \cos x)$

(d) $$\frac{(3x^2 + x)\left(\frac{1}{x}\right) - \log_e x \times (6x + 1)}{(3x^2 + x)^2}$$ (*quotient rule*)
$$= \frac{(3x + 1) - \log_e x \times (6x + 1)}{(3x^2 + x)^2}$$

(e) $$\frac{(2 + \sin x) \times (-3\sin x) - 3\cos x \times (\cos x)}{(2 + \sin x)^2}$$
$$= \frac{-6\sin x - 3\sin^2 x - 3\cos^2 x}{(2 + \sin x)^2}$$ (*quotient rule*)
$$= \frac{-3(2\sin x + 1)}{(2 + \sin x)^2}$$

(f) $$\frac{(2 + \cos x) \times (1) - x \times (-\sin x)}{(2 + \cos x)^2}$$ (*quotient rule*)
$$= \frac{\cos x + 2 + x\sin x}{(2 + \cos x)^2}$$

(If you are unsure of the answers to Questions R3 or R6, consult the relevant entries in the *Glossary*.)

Text answers

T1 The box representations of the functions are shown in Figure 8.

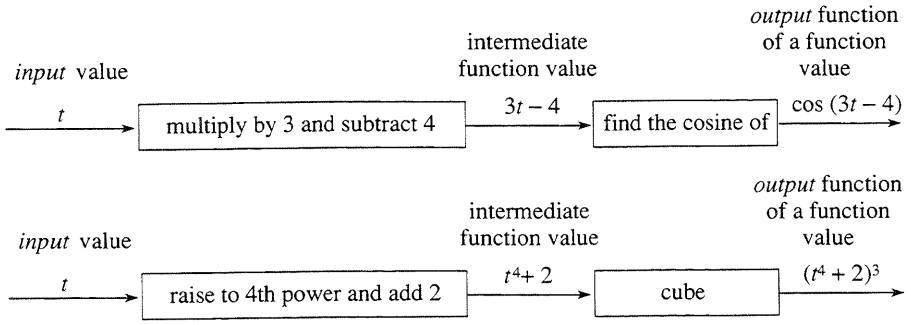


Figure 8 See Answer T1.

T2 (a) Composite; $g(x) = u = 2x$, $f(u) = e^u$

(b) Not composite; F is a product.

(c) Composite; the log *of* x cubed.
$u = g(x) = x^3$, $f(u) = \log_e u$

(d) Composite; $u = g(x) = x^2 + 2x + 1$, $f(u) = \sin u$

(e) Composite; the square *of* $\sin x$.
$u = g(x) = \sin x$, $f(u) = u^2$. (It is also a product.)

(f) Composite; $u = g(x) = \sin x$, $f(u) = u^3$. (It is also a product.)

(g) Composite; $u = g(x) = \cos x$, $f(u) = u^{0.5}$.

T3 $f(x)$ is defined only for $x \geq -1$; $g(x)$ is defined only for $x > 0$.

$f(g(x))$ defined only for those x for which $\log_e x > -1$, i.e. $x > 1/e$.

$g(f(x))$ always defined if $x > -1$, for then $f(x) > 0$.

T4 (a) $dq/dx = 2(2x - 3) \times 2 = 4(2x - 3)$

(b) $dq/dx = 2(3x^4 + 1) \times 12x^3 = 24x^3(3x^4 + 1)$

(c) $dq/dx = 2(x + \sin x) \times (1 + \cos x)$

T5 (a) $2e^{2x}$

(b) $2x\exp(x^2)$

(c) $5(x^2 + 1)^4 \times 2x = 10x(x^2 + 1)^4$

(d) $6\cos(6x)$

(e) $-3\sin(3x)$

(f) $\dfrac{1}{e^x} \times e^x = 1$

(g) $2x/(1 + x^2)$

(h) $-\omega\sin(\omega t + \phi)$

T6 $5y^4\dfrac{dy}{dx} = \cos x$, so $\dfrac{dy}{dx} = (\cos x)/5y^4$

T7 (a) $y = \arcsin(x^2)$, so $\sin y = x^2$

so, differentiating both sides gives us

$$\cos y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{\cos y} = \frac{2x}{\sqrt{1 - \sin^2 y}} = \frac{2x}{\sqrt{1 - x^4}}$$

Note that we have taken the positive square root expression for $\cos y$, as in the text, to ensure that dy/dx has the required sign.

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(b) There are no (real) values of x for which $\arccos(x^2+4)$ is meaningful. The function $\arccos u$ is only defined when its argument u is in the range $-1 \le u \le 1$. No x will produce an argument in this range. *Moral*: don't just differentiate without thinking first!

(c) $y = \arctan(3-x)$, $\quad \tan y = 3 - x$

therefore $\sec^2 y \dfrac{dy}{dx} = -1$

therefore $\dfrac{dy}{dx} = \dfrac{-1}{\sec^2 y} = \dfrac{-1}{1+\tan^2 y} = \dfrac{-1}{1+(3-x)^2}$

$$= \frac{-1}{10 - 6x + x^2}$$

T8 (a) $\dfrac{dx}{d\theta} = -a\sin\theta, \quad \dfrac{dy}{d\theta} = b\cos\theta$

so, $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta} = \dfrac{-b\cos\theta}{a\sin\theta} = \dfrac{-b}{a}\cot\theta$

(b) $\dfrac{dx}{dt} = 2at, \quad \dfrac{dy}{dt} = 2a, \quad \dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

(c) $\dfrac{dx}{dt} = c, \quad \dfrac{dy}{dt} = \dfrac{-c}{t^2}, \quad \dfrac{dy}{dx} = \dfrac{-1}{t^2}$

T9 (a) $f'(x) = 6x^5 - 4x, \quad f''(x) = 30x^4 - 4$

(b) $f'(x) = a, \quad f''(x) = 0$

(c) $f'(x) = a\cos x - b\sin x$

$f''(x) = -a\sin x - b\cos x$

Note that $f''(x) = -f(x)$, in this example.

(d) $f'(x) = 3e^{3x}$, $f''(x) = 9e^{3x}$

(e) $f'(x) = \dfrac{2x}{x^2+1}$

$$f''(x) = \frac{(x^2+1)\times 2 - 2x\times(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

(f) This is exactly the same as parts (a) to (e) except that $f'(x)$ is replaced by dy/dx and $f''(x)$ is replaced by $\dfrac{d^2y}{dx^2}$ in each case.

T10 (a) $v(t) = dx/dt = p + 2qt$, i.e. $\dot{x}(t) = p + 2qt$;

$a(t) = d^2x/dt^2 = 2q$, i.e. $\ddot{x}(t) = 2q$.

(b) When $x(t) = A\sin(\omega t)$, $\dot{x}(t) = A\omega\cos(\omega t)$ and

$\ddot{x}(t) = -A\omega^2\sin(\omega t)$.

When $x(t) = B\cos(\omega t)$, $\dot{x}(t) = -B\omega\sin(\omega t)$ and

$\ddot{x}(t) = -B\omega^2\cos(\omega t)$

Note that both these expressions for x satisfy the equation

$\ddot{x} = -\omega^2 x$.

T11 (a) $\dfrac{dy}{dx} = f'(x) = 20e^{5x}, \quad \dfrac{d^2y}{dx^2} = f''(x) = 100e^{5x}$

$\dfrac{d^3y}{dx^3} = f'''(x) = 500e^{5x}, \quad \dfrac{d^4y}{dx^4} = f^{(iv)}(x) = 2500e^{5x}$

$\dfrac{d^ny}{dx^n} = f^{(n)}(x) = 4\times 5^n \times e^{5x}$

(b) $\dfrac{dy}{dx} = 9x^2, \quad \dfrac{d^2y}{dx^2} = 18x, \quad \dfrac{d^3y}{dx^3} = 18$

$\dfrac{d^4y}{dx^4} = 0, \quad \dfrac{d^ny}{dx^n} = 0 \quad$ for $\quad n \ge 4$

(c) $\dfrac{dy}{dx} = -2A\sin(2x) + 2B\cos(2x)$

$\dfrac{d^2y}{dx^2} = -4A\cos(2x) - 4B\sin(2x)$

$\dfrac{d^3y}{dx^3} = 8A\sin(2x) - 8B\cos(2x)$

$\dfrac{d^4y}{dx^4} = 16A\cos(2x) + 16B\sin(2x)$

Note that $\dfrac{d^2y}{dx^2} = -2^2 y$ and that $\dfrac{d^4y}{dx^4} = 2^4 y$

If n is even $\dfrac{d^ny}{dx^n} = (-1)^{n/2}(2)^n y$,

if n is odd $\dfrac{d^ny}{dx^n} = (-1)^{(n-1)/2}(2)^{n-1}\dfrac{dy}{dx}$

(d) $\dfrac{dy}{dx} = \dfrac{1}{x}, \quad \dfrac{d^2y}{dx^2} = \dfrac{-1}{x^2}, \quad \dfrac{d^3y}{dx^3} = \dfrac{2}{x^3}$

$\dfrac{d^4y}{dx^4} = \dfrac{-6}{x^4}, \quad \dfrac{d^ny}{dx^n} = \dfrac{(-1)^{n-1}(n-1)(n-2)\ldots 2}{x^n}$

T12 (You may wish to look back at Answer T8 for some of the results used in this answer.)

(a) $\dfrac{dx}{d\theta} = -\sin\theta, \quad \dfrac{dy}{d\theta} = \cos\theta$ therefore

$\dfrac{dy}{dx} = -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}(-\cot\theta)/(-\sin\theta) = \operatorname{cosec}^2\theta/(-\sin\theta)$

$$= \frac{1}{\sin^2\theta}\times\left(-\frac{1}{\sin\theta}\right) = -\frac{1}{\sin^3\theta} = -\operatorname{cosec}^3\theta$$

(b) $\dfrac{dx}{d\theta} = -a\sin\theta, \quad \dfrac{dy}{d\theta} = b\cos\theta$ therefore

$\dfrac{dy}{dx} = \dfrac{-b\cos\theta}{a\sin\theta} = \dfrac{-b}{a}\cot\theta$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(-\dfrac{b}{a}\cot\theta\right)\Big/(-a\sin\theta)$

$$= \frac{b}{a}\operatorname{cosec}^2\theta\Big/(-a\sin\theta)$$

$$= \frac{b}{a}\frac{1}{\sin^2\theta}\times\left(-\frac{1}{a\sin\theta}\right) = -\frac{b}{a^2\sin^3\theta}$$

$$= \frac{-b}{a^2}\operatorname{cosec}^3\theta$$

(c) $\dfrac{dx}{dt} = c, \quad \dfrac{dy}{dt} = -\dfrac{c}{t^2}$ therefore $\dfrac{dy}{dx} = -\dfrac{1}{t^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(-\dfrac{1}{t^2}\right)\Big/c = +\dfrac{2}{ct^3}$

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Exit test answers

E1 (a) Function of a function; read as 'sin *of* $3x$'.

(b) Function of a function; read as 'log *of* $(1-x^2)$'. We require that

$1-x^2>0$ so that $x^2<1$, i.e. $-1<x<1$.

(c) If it is rewritten as $y=x^{1/3}$; it is a function, read as cube root *of* x so not a function of a function.

(Reread Subsection 2.1 if you had difficulty with this question.)

E2 (i) (a) $3\cos(3x)$, (b) $\dfrac{-2x}{1-x^2}$, (c) $\dfrac{1}{3x^{2/3}}$

(ii) $f(g(x))=5(x^3+2)+4=5u+4$

$\dfrac{du}{dx}=3x^2, f'(u)=5$, therefore using the chain rule, the derivative $\dfrac{df}{du}$ is $5\times 3x^2=15x^2$.

$g(f(x))=(5x+4)^3+2$, therefore the derivative is

$$\frac{dg}{dx}=3(5x+4)^2\times 5=15(5x+4)^2$$

(Reread Subsection 2.2 if you had difficulty with this question.)

E3 $\cos y=3x-2, \quad -\sin y\dfrac{dy}{dx}=3$

so $$\frac{dy}{dx}=\frac{-3}{\sin y}=\frac{-3}{\sqrt{1-\cos^2 y}}=\frac{-3}{\sqrt{1-(3x-2)^2}}$$

$$=\frac{-3}{\sqrt{-9x^2+12x-3}}$$

Note that we have expressed $\sin y$ in terms of a positive square root to ensure that dy/dx is negative, as is required of the arccos function.

(Reread Subsection 2.3 if you had difficulty with this question.)

E4 $\dfrac{dx}{dt}=3\sqrt{2}, \quad \dfrac{dy}{dt}=3\sqrt{2}-9.8t$

therefore $$\frac{dy}{dx}=\frac{3\sqrt{2}-9.8t}{3\sqrt{2}}$$

(Reread Subsection 2.4 if you had difficulty with this question.)

E5 $y=\sin(3x), \quad \dfrac{dy}{dx}=3\cos(3x)$

therefore $\dfrac{d^2y}{dx^2}=-9\sin(3x)$ and $1\geq 3x\geq -1$

$y=\log_e(1-x^2), \quad \dfrac{dy}{dx}=\dfrac{-2x}{1-x^2}$

therefore $$\frac{d^2y}{dx^2}=\frac{(1-x^2)(-2)-(-2x)(-2x)}{(1-x^2)^2}$$

$$=\frac{-2(1-x^2+2x^2)}{(1-x^2)^2}=\frac{-2(1+x^2)}{(1-x^2)^2}$$

and $(1-x^2)\geq 0$ so $|x|\leq 1$

$y=x^{1/3}, \quad \dfrac{dy}{dx}=\dfrac{1}{3}x^{-2/3}$

therefore $$\frac{d^2y}{dx^2}=\frac{-2}{9x^{5/3}}$$

$\cos y=3x-2, \quad \dfrac{dy}{dx}=\dfrac{-3}{(-9x^2+12x-3)^{1/2}}=f(g(x))$

and $1\geq 3x+2\geq -1$

where $g(x)=u=-9x^2+12x-3$ $f(u)=\dfrac{-3}{u^{1/2}}$

$$\frac{du}{dx}=-18x+12, \quad f'(u)=\frac{3}{2u^{3/2}}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2(-9x^2+12x-3)^{3/2}}\times(-18x+12)$$

$$=\frac{-27x+18}{(-9x^2+12x-3)^{3/2}}$$

$x=2+3\sqrt{2}t, \quad y=1+3\sqrt{2}t-4.9t^2$

$$\frac{dy}{dx}=\frac{3\sqrt{2}-9.8t}{3\sqrt{2}}, \quad \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{3\sqrt{2}-9.8t}{3\sqrt{2}}\right)\Big/3\sqrt{2}$$

$$=\frac{-9.8}{3\sqrt{2}}\times\frac{1}{3\sqrt{2}}=\frac{-9.8}{18}$$

(Reread Subsections 3.1 and 3.3 if you had difficulty with this question.)

E6 (a) $f'(x)=6x^5-8x^3-3x^2+6$

$f''(x)=30x^4-24x^2-6x$

$f'''(x)=120x^3-48x-6$

$f^{(iv)}(x)=360x^2-48$

$f^{(v)}(x)=720x$

$f^{(n)}(x)=0, \quad n>6.$

(b) $f'(x)=xe^x+1e^x=(x+1)e^x$

$f''(x)=(x+1)e^x+1e^x=(x+2)e^x$

$f'''(x)=(x+2)e^x+1e^x=(x+3)e^x$

$f^{(iv)}(x)=(x+4)e^x$

$f^{(v)}(x)=(x+5)e^x$

$f^{(n)}(x)=(x+n)e^x$, for all n.

(Reread Subsection 3.2 if you had difficulty with this question.)

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