Basic differentiation

Module M4.2

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1 Opening items

1.1 Module introduction

Being able to *differentiate* accurately and confidently is a necessary skill for any physicist. Differentiation is such a powerful tool with so many applications that it is worth some effort to ensure that you have mastered the techniques explained in this module.

The main objective of this module is to answer the following question:

> 'If we know how to differentiate two functions $f(x)$ and $g(x)$, how do we differentiate their sum, difference, product and quotient?'

To answer this question we discuss first the idea of a *function* and how functions can be combined to produce new ones. Section 2 examines the mathematical basis of *differentiation*, giving first a graphical interpretation and then a formal definition.

Section 3 answers our question and is the core of the module. It provides a list of derivatives of various standard functions (such as power functions (x^n) and trigonometric functions) and introduces the various rules that will enable you to differentiate a wide range of combinations of these standard functions. In particular it will explain how to differentiate *sums*, *constant multiples*, *products* and *quotients* of the standard functions. Sometimes the method is straightforward, as with the derivative of a *sum*, but it may also be quite complicated, as with the derivative of a *quotient*.

We also discuss the *logarithmic* and *exponential* functions, and, while $\log_e x$ and e^x are the most important of such functions, it is sometimes necessary to be able to differentiate $\log_a x$ and a^x where a is some positive number other than e. In Section 4 we consider why e^x is so important and then add a^x and $\log_a x$ to the list of functions that can be differentiated.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 6. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 Differentiate each of the following functions (it is not necessary to simplify your answers).

(a) $f(x) = x^2 \sin x + \log_e x$

(b) $f(x) = (x^3 - 6x)(2x^2 + 5x - 1)$ (use the *product rule*)

(c) $f(x) = \dfrac{1}{\sin x}$

(d) $f(x) = \dfrac{x^2 + 4x - 1}{x^2 + 9}$

(e) $f(x) = \dfrac{x \tan x}{3x^2 - 2x + 1}$

(f) $f(x) = 2^x + x^2 + e^2$

(g) $f(x) = (e^x + e^{-x})^2$

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Question F2 If at time t the displacement, $s_x(t)$, of a particle along the x-axis from some particular reference point is given by

$$s_x(t) = e^{-\alpha t}[A\cos(\omega t) + B\sin(\omega t)]$$

where A, B, α and ω are constants, find expressions for $v_x(t)$, the velocity at time t, and $a_x(t)$ the acceleration at time t, and show that

$$a_x(t) + 2\alpha v_x(t) + (\omega^2 + \alpha^2)s_x(t) = 0$$

1.3 Ready to study?

Study comment To begin the study of this module you will need to be familiar with the following terms: *base* (of a *logarithm*), *exponential function*, *inequality* (in particular *greater than* (>) and *less than* (<)), *integer*, *inverse function*, *logarithmic function*, *modulus* (or *absolute value*, $|x|$), *product*, *quotient*, *radian*, *real number*, *reciprocal*, *set*, *square root*, *sum*, *trigonometric function* and *trigonometric identities*. (The *trigonometric identities* that you require are repeated in this module, as is the definition of the *modulus* function.) In addition you will need to have some familiarity with the concept of a *function*, and with related terms such as *argument*, *codomain*, *domain* and *variable* (both *dependent* and *independent*), but the precise meaning of these terms is briefly reviewed in the module. Similar comments apply to physics concepts used in this module, such as *acceleration*, *displacement*, *force*, *speed*, *velocity* and *Newton's second law*. If you are uncertain of any of these items you can review them now by referring to the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics, or to improve your general algebraic skills, before embarking on this module.

Question R1 Write the following sums, products and quotients of logarithms and exponentials as compactly as you can:

(a) $\log_{10} x + \log_{10} y$, (b) $(\log_2 4 \times \log_2 4)^3$, (c) $\log_e x - \log_e y$, (d) e^x/e^y,

(e) $(e^x/e^y)^2$, (f) $\log_e x + \log_e x + \log_e x$.

You will find the answers to these questions in Section 6.

Question R2 If $f(x) = \cos x$, $g(x) = \sin x$, and $h(x) = \tan x$, rewrite the following expressions in terms of trigonometric functions and simplify them where possible:

(a) $[f(x) + g(x)]^2$

(b) $f(x)h(x) + g(x)/h(x)$

(c) $[f(x) - g(x)]/[1 + h(x)]$

The notations $\sin x$ and $\sin(x)$ are identical in meaning. On the whole we will follow customary practice and omit the brackets in this module unless the argument of the function consists of something more than a single letter. Similar comments apply to other functions, such as $\tan(x)$ and $\log_e(x)$.

Question R3 Given that $f(x) = \dfrac{x^2}{4} - 1$, find the following values, $f(1)$, $f(0)$, $f(-2)$, $|f(0)|$, $|f(0) - f(-2)|$ and sketch the graph of $y = f(x)$

Question R4 Given that a^x and $\log_a x$ are inverse functions, simplify the following:

(a) $\log_e[e^{2\log_{10}(x)}]$, (b) $\exp(\log_e(10^x))$, (c) $\log_2(2^2)$.

On some calculators $\log_e$ corresponds to the ln key.

2 Variables, functions and derivatives

2.1 Functions and variables

Study comment An understanding of *functions* is crucial to an understanding of differentiation, and it is vital that the notation and terminology used to describe functions should be clear and unambiguous. For that reason, this subsection reviews the definitions of terms such as *function* and *variable* even though it is assumed that you have met these ideas before. If you are completely unfamiliar with these concepts you should consult the entry on *functions* in the *Glossary*.

A *function* f is a rule that assigns a single value $f(x)$ in a set called the *codomain* to each value x in a set called the *domain*.

Functions are very often defined by formulae, for example $f(x) = x^2$, and in such cases we assume, unless we are told otherwise, that the domain is the largest set of real values for which the formula makes sense. In the case of $f(x) = x^2$ the domain of

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the function is the set of all real numbers. The function $g(x) = \dfrac{1}{1-x}$ is not defined when $x = 1$, since 1/0 has no meaning, and so we take the set of all real numbers x with $x \neq 1$ as its domain.

The function $f(x) = 1 + x + x^2$ (1a)

is another example of a function that is defined for all values of x. One may think of the function as a sort of 'machine' with x as the input and $1 + x + x^2$ as the output. The input x is known as the *independent variable* and, if we write $y = f(x)$, the output y is known as the *dependent variable* since the function $f(x)$ determines the way in which y depends on x.

It is important to note that the same function f could equally well be defined using some other symbol, such as t, to represent the independent variable:

$$f(t) = 1 + t + t^2 \qquad (1b)$$

This freedom to relabel the independent variable is often of great use, though it is vital that such changes are made consistently throughout an equation.

We may *evaluate* the function in Equation 1, whether we call it $f(x)$ or $f(t)$, for any value of the independent variable; for example, if we choose to use x to denote the independent variable, and set $x = 1$, we have

$$f(1) = 1 + 1 + 1^2 = 3$$

Similarly, if $x = \pi$ $\quad f(\pi) = 1 + \pi + \pi^2$

and, if $x = 2a$ $\quad f(2a) = 1 + (2a) + (2a)^2 = 1 + 2a + 4a^2$

We may even apply the function to $2x$, $-x$, or $1 + x$ to obtain

$$f(2x) = 1 + (2x) + (2x)^2 = 1 + 2x + 4x^2$$

$$f(-x) = 1 + (-x) + (-x)^2 = 1 - x + x^2$$

$$f(1 + x) = 1 + (1 + x) + (1 + x)^2 = 3 + 3x + x^2$$

When we write expressions such as $f(\pi)$ or $f(2a)$, whatever appears within the brackets is called the *argument* of the function. The value of $f(x)$ is determined by the *value* of its argument, irrespective of what we call the argument.

A special note about $\sqrt{x}$ Generally in *FLAP* we follow the convention that $\sqrt{x}$ may be positive or negative. Thus $\sqrt{4} = \pm 2$. A consequence of this convention is that $f(x) = \sqrt{x}$ does not define a function since it does not associate a unique value of $f(x)$ with each value of x. This would be an exceptionally inconvenient convention to follow in this module so instead we adopt the convention that $\sqrt{x}$ is positive. Of course, it remains true that the square roots of x may be positive or negative, $\sqrt{x}$ or $-\sqrt{x}$, so within this convention we should call $\sqrt{x}$ the positive square root of x, and not simply the square root. In a similar spirit $x^{1/2} = \sqrt{x}$ will also be a positive quantity. □

◆ If $f(x) = \cos^2 x + \sin x$ find $f(2x)$ and $f\left(\dfrac{\pi}{4}\right)$ (☞)

◇ $f(2x) = \cos^2 2x + \sin 2x$

and $f\left(\dfrac{\pi}{4}\right) = \cos^2\dfrac{\pi}{4} + \sin\dfrac{\pi}{4} = \left(\dfrac{1}{\sqrt{2}}\right)^2 + \dfrac{1}{\sqrt{2}} = \dfrac{1}{2} + \dfrac{\sqrt{2}}{2} = \dfrac{1+\sqrt{2}}{2}$ (☞) □

Note that $(\cos x)^2$ and $\cos^2 x$ have the same meaning, and while the latter is often the more convenient form, we may occasionally use the former for added emphasis.

Note that: $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Question T1 If $f(x) = x^2 + 1$ and $g(x) = 2x$,

(a) write down expressions for $f(\sqrt{x})$ and $g\left(\dfrac{x}{2}\right)$.

(b) For which values of the independent variable x is $f(x) = g(x)$?

(c) For which value of the independent variable x is the following true?

$$\frac{f(x + 0.1) - f(x)}{0.1} = 0.2 \quad \square$$

You will find the answers to all *Text questions* in Section 6.

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2.2 Rates of change, gradients and derivatives

gradient
chord
tangent
limit

The graph of the function $g(t) = t^2$ is shown in Figure 1. It is clear from the graph that as the value of t increases from $t = 0$, the value of $g(t)$ also increases from 0. In particular, note that as t increases from 0 to 1 the value of $g(t)$ changes from 0 to 1, and that as t increases from 1 to 2 the value of $g(t)$ changes from 1 to 4. Thus, the change in the value of $g(t)$ that corresponds to a change of one unit in the value of t depends on the initial value of t. For this particular function, if the initial value of t is large, then the change in $g(t)$ will also be large; for example, the change in $g(t)$ from $t = 100$ to $t = 101$ is

$$g(101) - g(100) = 101^2 - 100^2 = 201$$

Clearly, as t increases the *rate* at which $g(t)$ is changing is itself changing, becoming greater and greater. How are we to measure the rate at which a function changes? If you imagine yourself walking up a hill in the shape of the graph shown in Figure 1, starting at $t = 0$ and travelling to the right, your path would become steeper and steeper. It would become increasingly difficult to make progress, because, with each step you took, the slope would become greater and your walk would become more and more of a climb. The *slope* of the graph is the key; once we can describe the slope precisely for any value of t, we will be able to measure the *rate of change* of the function $g(t)$ — or of any other function.

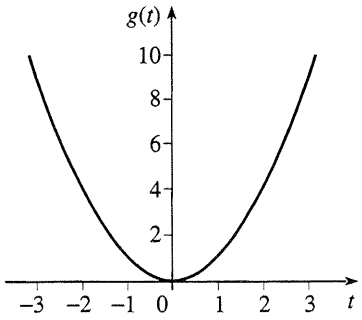
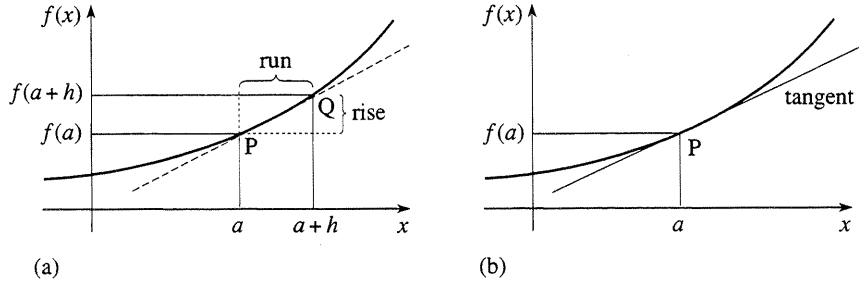


Figure 1 The graph of $g(t) = t^2$.

The graph of an arbitrary function $f(x)$ is shown in Figure 2a. Roughly speaking, the slope of this graph at the point P, where $x = a$ and $f(x) = f(a)$, is the same as the slope of the dashed straight line PQ where Q is a nearby point corresponding to $x = a + h$ and $f(x) = f(a + h)$. Of course, the two slopes are not exactly the same; the point P is fixed and its location determines the slope of $f(x)$ at P, whereas the slope of the line PQ will depend on the location of Q as well as that of P. Nonetheless, the two slopes are similar, so we can roughly describe the slope of the curve in terms of the **gradient** of the line PQ which is defined by

$$\text{gradient of the line PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad (2)(\text{☞})$$



Now, the line PQ (the dashed line in Figure 2a) that cuts the graph of $f(x)$ at P and Q and is called a **chord**, but if we let the point Q get closer and closer to P then this chord becomes more and more like the **tangent** in Figure 2b that just touches the graph at P. It is the gradient of this tangent that really represents the slope of the curve at P, this is clear from the figure and from the fact that the gradient of the tangent (unlike that of the chord) is determined by the location of P alone. Fortunately, the gradient of the tangent is fairly easy to work out, all we have to do is to consider what happens to the gradient of the chord PQ as Q gets closer and closer to P. As Q approaches P, h gets smaller and the gradient of the chord PQ approaches the gradient of the tangent at P ever more closely. Expressed more formally, the gradient of the tangent at P is given by Equation 2 in the **limit** as h tends to zero. Thus

$$\text{gradient of tangent at P} = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] \qquad (3)$$

Since this gradient represents the slope of the graph of $f(x)$ at P, it makes sense to *define* the gradient of the graph of $f(x)$ at P to be the gradient of its tangent at P. We call this gradient the *derivative* of $f(x)$ at $x = a$ and denote it by $f'(a)$ so that

$$f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] \qquad (4)$$

It is quite common to use Δx in place of h to indicate a small change in the variable x. (Δ is the uppercase Greek letter delta.) Moreover, if we let $y = f(x)$ then we can use Δy to represent $f(a + h) - f(a)$, and we can write

$$\text{gradient of the line PQ} = \frac{\Delta y}{\Delta x}$$

Figure 2 The graph of an arbitrary function.

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rate of change
derived function
derivative

The derivative at $x = a$ defined in this way also represents the **rate of change** of $f(x)$ with respect to x when $x = a$. This last point is even easier to appreciate if we introduce a dependent variable y such that $y = f(x)$, for we can the represent the vertical *rise* in Figure 2a by $\Delta y = f(a + h) - f(a)$, and the horizontal *run* by $\Delta x = h$. With these definitions it follows from Equation 4 that the derivative at $x = a$ is given by

$$f'(a) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

Even greater emphasis can be given to the idea that the derivative is a rate of change with respect to x by replacing $f'(a)$ by the alternative symbol $\frac{dy}{dx}(a)$.

$\frac{dy}{dx}$ is read as 'dee y by dee x'.

$$\text{Thus, } \frac{dy}{dx}(a) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \qquad (5)$$

The definition of the derivative at $x = a$ represented by Equations 4 and 5 can be applied at any point on the graph provided that Δy and Δx can be defined at that point and that their quotient $\Delta y/\Delta x$ has a unique limit as $\Delta x \to 0$ at that point. Thus, provided the derivative of $f(x)$ exists at every point within some domain (i.e. some set of x values) it is possible to define a new function on that domain that associates any given value of x with the gradient of $f(x)$ at that point. This new function is called the **derived function** or **derivative** of $f(x)$ and is written $f'(x)$ or $\frac{df}{dx}(x)$. If $y = f(x)$, the derived function may also be written $\frac{dy}{dx}(x)$ or just $\frac{dy}{dx}$. (☞) Thus, provided unique limits exist

Although this terminology may blurr the important distinction between a function and the value of the function at a particular point, in what follows we will often refer to derived functions as derivatives and represent them by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \qquad (6)$$

When using this formula it is important to remember that dy/dx is *not* a quotient of two quantities dy and dx even though it may look like one. At a given value of x, the derivative dy/dx represents the gradient of the graph of $y = f(x)$ at that value of x.

Although the graphical interpretation of the derivative is important, it does not lend itself to calculation since drawing tangents can only be approximate, and in any case it would be impossible to do it for every point on the graph. However, in simple cases it is not difficult to determine the derivative from the formula for the function and the above definition (Equation 6). Consider the case of $f(x) = x^2$, for which

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] = \lim_{h \to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] = \lim_{h \to 0}\left(\frac{2xh + h^2}{h}\right)$$

i.e. $f'(x) = \lim_{h \to 0}(2x + h) = 2x$

so that $f'(x) = 2x$

The function $f(x) = x^2$ has the set of all real numbers as its domain, and its derivative $f'(x) = 2x$ has the same domain.

As a further example, we can use the definition to obtain the derivative of the function $f(x) = 1/x$.

In this case $f(x + h) = \frac{1}{x + h}$

giving us

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{1}{h}\left(\frac{x - (x+h)}{(x+h)x}\right) = \frac{-1}{(x+h)x}$$

As h tends to zero the expression $\frac{-1}{(x+h)x}$ approaches $-\frac{1}{x^2}$.

So, $$f'(x) = \lim_{h \to 0}\left[\frac{-1}{(x+h)x}\right] = -\frac{1}{x^2} \qquad (7)$$

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The function $f(x) = 1/x$ has the set of all non-zero real numbers as its domain, and its derivative $f'(x) = -1/x^2$ has the same domain.

Question T2 (a) If $f(x) = 1/(\omega x)$ where ω is a constant, use the definition of the derivative to find $f'(x)$.

(b) If $g(t) = 1/(\omega t)$ use the answer to part (a) to write $g'(t)$ and $g'(2t)$. □

Don't be misled into thinking that if a function exists then its derivative must also exist. The function $f(x) = |x|$ ('the *modulus* of x') illustrates the situation where the derivative has a smaller domain than the function (see Figure 3). The modulus of x is defined by

$$|x| = x \quad \text{if } x > 0$$

and $$|x| = -x \quad \text{if } x < 0$$

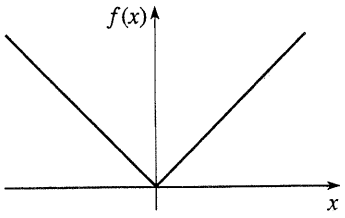


Figure 3 The graph of $f(x) = |x|$.

For positive x the graph of $f(x) = |x|$ is a straight line with gradient 1, so

$$f'(x) = 1 \text{ if } x > 0$$

while for negative x the graph is a straight line with gradient -1,

so $$f'(x) = -1 \text{ if } x < 0$$

However, when $x = 0$ no unique limit exists since we get different values for the limit depending on how we calculate it. In particular, if we approach $x = 0$ from the positive side, where $x > 0$ and h is positive

$$\lim_{h\to 0+}\left[\frac{f(0+h)-f(0)}{h}\right] = \lim_{h\to 0+}\left(\frac{|h|-|0|}{h}\right) = \lim_{h\to 0+}\left(\frac{|h|}{h}\right) = 1$$

whereas, if we approach $x = 0$ from the negative side, where $x < 0$ and h is negative

$$\lim_{h\to 0-}\left[\frac{f(0+h)-f(0)}{h}\right] = \lim_{h\to 0-}\left(\frac{|h|-|0|}{h}\right) = \lim_{h\to 0-}\left(\frac{|h|}{h}\right) = -1$$

So, if h approaches zero through positive values the limit is 1, but if h approaches zero through negative values the limit is -1. A *unique* limit exists only if we obtain the *same* result no matter how h approaches 0. Since *different* answers have been obtained in this case there is no unique limit and hence no derivative at $x = 0$. Although the original function is defined for all real values of x the derivative is only defined for non-zero values of x.

On the whole, the domains of functions are of more concern to mathematicians than to physicists. Nonetheless, it is important that you should be aware of their significance, and prepared to investigate them if necessary.

Question T3 (a) Use your calculator to convince yourself that $\dfrac{\exp(h)-1}{h}$ is very nearly equal to 1 when h is very small.

(b) Let $f(x) = \exp(x)$. What does part (a) tell you about $f'(0)$?

(c) Use the fact that $\lim_{h\to 0}\left[\dfrac{\exp(h)-1}{h}\right] = 1$, and the definition of the derivative to show that $f'(x) = \exp(x)$. □

$\exp(x)$ is often written in the alternative form e^x where e = 2.718 281 828 46 to 11 decimal places. When written in this form it is easy to see that $\exp(0) = 1$ and that $\exp(x + h) = \exp(x)\exp(h)$.

2.3 Notation for derivatives

Various notations are in common use for the derivative of a given function. Two such notations have already been introduced in this module and you will probably meet a third elsewhere. Here we provide a brief summary.

Function notation

This is the notation we have mainly used so far, in which a function is represented by f or $f(x)$ or some similar symbol and its derivative (the derived function) is represented by f' or $f'(x)$. This is a neat and compact notation but care is needed in handwritten work to make sure that the all-important prime (′) is clearly visible.

Leibniz notation

This is the dy/dx notation that is especially popular among physicists. We will make much use of it in what follows.

Gottfried Wilhelm Leibniz (1646–1716) was a philosopher and mathematician and one of the co-inventors of *calculus*.

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If we let $y = f(x)$, then in Leibniz notation the derivative is written $\frac{dy}{dx}(x)$ or $\frac{df}{dx}(x)$. Sometimes these are abbreviated to $\frac{dy}{dx}$ (in which case it is assumed that we are regarding the variable y as a function $y(x)$ of x), or $\frac{df}{dx}$ (in which case it is assumed that we are discussing a function $f(x)$).

differentiation

This notation developed from the use of Δx and Δy to represent small increments in the variables x and y, respectively. Leibniz intended dy/dx to represent the ratio $\Delta y/\Delta x$ when both increments were 'infinitesimally small'. (The concept of a limit was not developed until much later.)

An advantage of Leibniz notation is that it is also possible to write the derivative as

$$\frac{d}{dx}[f(x)]$$

so that for a specific function, $f(x) = x^2$ say, we could write

$$\frac{d}{dx}(x^2) = 2x$$

Newtonian notation

A different notation that is especially common in mechanics, but which will not be used in this module, uses a dot to denote derivatives. Thus, if $x(t)$ represents the x coordinate of a moving particle at time t, then $\dot{x}(t)$ represents the rate of change of x with respect to time, i.e. the x component of the particle's velocity.

Isaac Newton (1642–1727), co-inventor of *calculus* and founder of Newtonian mechanics, is widely regarded as the genius of his age.

◆ Given that $y = f(x) = 1/x$, and recalling Equation 7, determine the following:

(a) $\frac{dy}{dx}(2)$, (b) $\frac{df}{dx}(2a)$, (c) $\frac{d}{dt}\left(\frac{1}{t}\right)$, (d) $f'(2+3x)$.

◇ We know from Equation 7 that if $f(x) = 1/x$, then $f'(x) = -\frac{1}{x^2}$, so that

(a) $\frac{dy}{dx}(2) = -\frac{1}{2^2} = -\frac{1}{4}$ (b) $\frac{df}{dx}(2a) = -\frac{1}{(2a)^2} = -\frac{1}{4a^2}$

(c) $\frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$ (d) $f'(2+3x) = -\frac{1}{(2+3x)^2}$ □

3 Derivatives of simple functions

Now that the definitions are out of the way we can get down to the real business of *doing* calculus. The procedure by which derivatives are determined is called **differentiation**. In practice everybody who uses differentiation regularly, knows the derivatives of various standard functions (x, $\sin x$, $\exp(x)$, etc.) and knows some simple rules for finding the derivatives of various combinations (sums, differences, products, quotients and reciprocals) of those standard derivatives. The formal definition of a derivative is rarely used in practice. This section introduces the standard derivatives and the basic rules for combining them.

3.1 Derivatives of basic functions

Tables 1a and 1b list a number of functions with which you should already be familiar along with their derivatives. Each of these derivatives can be deduced from the definition given in the last section, though the proof is not always easy. If you are going to use calculus frequently you will need to know these derivatives or at least know where you can look them up quickly. When using the tables it is important to remember the following points:

- n and k are constants.
- The functions in Table 1b are special cases of those in Table 1a, corresponding to $k = 1$.
- In each of the trigonometric functions x must be an angle in radians or a dimensionless real variable.

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Table 1 Some standard derivatives.

(a)

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx \cot kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

(b)

$f(x)$	$f'(x)$
1	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\exp(x)$	$\exp(x)$
$\log_e(x)$	$1/x$

Question T4 Use the definition of the derivative to show that if $f(x) = x^n$ and n is a positive integer, then $f'(x) = nx^{n-1}$. (☞) □

Answer T4 uses the *binomial theorem*. If you are not familiar with this just read the answer, do not attempt the question.

The best way to get to know the standard derivatives is to use them frequently. Here are a few questions to start the process of familiarization.

◆ Find the derivatives (using Table 1) of the following functions:

(a) $f(x) = x^3$, (b) $g(t) = 1/t^3$, (c) $h(x) = \dfrac{1}{\sqrt[5]{x^2}}$.

◇ In each case we express the function in the form x^n so that we may apply the rule discussed in Question T4.

(a) For $f(x) = x^3$ we have $n = 3$, so $f'(x) = 3x^{3-1} = 3x^2$.

(b) If we rewrite $g(t) = 1/t^3$ as $g(t) = t^{-3}$, we have $n = -3$, so

$g'(t) = (-3)t^{-3-1} = -3t^{-4} = -3/t^4$.

(c) If we rewrite $h(x) = \dfrac{1}{\sqrt[5]{x^2}}$ as $h(x) = \dfrac{1}{x^{2/5}} = x^{-2/5}$, we have $n = -\dfrac{2}{5}$ so

$$h'(x) = \left(-\frac{2}{5}\right)x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\frac{2}{5}\frac{1}{x^{7/5}} = -\frac{2}{5}\frac{1}{\sqrt[5]{x^7}}$$ □

◆ Which function in Table 1 is its own derivative?

◇ $y = \exp(x)$ has $\dfrac{dy}{dx} = \exp(x)$. (It is this property of the exponential function which makes it so important.) □

Question T5 Find where the gradients of the tangents to the graph of $y = \sin x$ have the following values: (a) 0, (b) −1, (c) +1. (☞) □

Remember that angles must be measured in *radians* if Table 1 is to apply.

Question T6 (a) Use your calculator (making sure that it is in radian mode) to investigate the limits

$$\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] \quad \text{and} \quad \lim_{h\to 0}\left[\frac{\sin h}{h}\right]$$

Using successively smaller values of h (e.g. $h = \pm 0.1, \pm 0.01$, etc.) try to estimate the limit in each case.

(b) Use the limits obtained in part (a) to find $\dfrac{d}{dx}(\sin x)$ from the general definition of the derived function.

(*Hint*: Use the trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$.) □

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Question T7 Which functions in Table 1b have derivatives which are:

(a) always positive;

(b) always negative;

(c) can be positive or negative? □

sum rule

3.2 Derivative of a sum of functions

While it is possible, in theory, to find the derivative of any given function from the definition, this would in fact be an arduous process. In practice, those using calculus employ a set of simple rules which can be applied to combinations of functions to find the derivatives of a wide variety of functions with relative ease. The first of these simple rules is called the **sum rule**, which states:

The derivative of a sum of functions is the sum of the derivatives of the individual functions.

If $f(x)$ and $g(x)$ are two functions, this can be written as:

sum rule $$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \qquad (8)$$

This is an important rule and should be committed to memory.

Alternatively we may write this rule in the form

$$[f(x)+g(x)]' = f'(x)+g'(x) \quad \text{or just} \quad (f+g)' = f' + g'$$

This rule enables us to differentiate functions such as $y = x^{1/2} + \log_e x$. To do so we first note that it is the sum of two standard functions $f(x) = x^{1/2}$ and $g(x) = \log_e x$, we then differentiate each of these functions and finally add the derivatives to obtain the required answer.

So, if $f(x) = x^{1/2}$ and $g(x) = \log_e x$

then, from Table 1

$$\frac{df}{dx} = \frac{1}{2}x^{-1/2} \quad \text{and} \quad \frac{dg}{dx} = \frac{1}{x}$$

and if $y = \sqrt{x} + \log_e x$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{x}$

Remember, in this module we are taking $\sqrt{x}$ to be positive.

It is usually a good idea to *think* before applying the rule, as in the following question.

◆ Find the derivative of $h(x) = \dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}$.

◇ The function $h(x)$ can be written as $h(x) = \cot x + \tan x$ and is therefore a sum of two of the standard functions that appear in Table 1.

Let $f(x) = \cot x$ to obtain $f'(x) = -\text{cosec}^2 x$.

Let $g(x) = \tan x$ to obtain $g'(x) = \sec^2 x$.

Since $h(x) = \cot x + \tan x$ we have $h'(x) = -\text{cosec}^2 x + \sec^2 x$. □

The rule 'derivative of a sum equals the sum of the derivatives' applies equally well if more than two functions are added together. This allows us to obtain the derivatives of functions such as those in the following question.

Question T8 Use the sum rule to find dy/dx in each of the following cases:

(a) $y = 2 + \sqrt[3]{x} + e^x$

(b) $y = (1 + \sqrt{x})^2$

(c) $y = \log_e(xe^x)$

(d) $y = \cot x \sin x + \tan x \cos x$.

(*Hint*: Write each function as a sum of functions appearing in Table 1.) □

Question T9 If you were to use the definition of the derived function (Equation 6) to show that the 'derivative of the sum is the sum of the derivatives', what assumption must you make about the limit of a sum? □

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3.3 Derivative of a constant multiple of a function

constant multiple rule

Table 1 gives the derivative of $\sin x$, for example, but what about $2\sin x$? In other words, what is the derivative of a constant times a function? The answer is given by the **constant multiple rule,** which states:

The derivative of a constant times a function is equal to the constant times the derivative of the function.

Using k for the constant and $f(x)$ for the function, this result can be written as:

constant multiple rule $$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)] \qquad (9)$$

This is an important rule and should be committed to memory.

i.e. $[kf(x)]' = kf'(x)$ or just $(kf)' = kf'$

Therefore, for the above example,

$$\frac{d}{dx}(2\sin x) = 2\frac{d}{dx}(\sin x) = 2\cos x$$

similarly, $$\frac{d}{dx}(\pi e^x) = \pi\frac{d}{dx}(e^x) = \pi \times e^x = \pi e^x$$

◆ Differentiate $h(x) = \left(x + \frac{1}{x}\right)^3$

◇ If we expand the function, we find

$$\left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2 \times \frac{1}{x} + 3x \times \frac{1}{x^2} + \frac{1}{x^3}$$

$$= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$$

$$= x^3 + 3x + 3x^{-1} + x^{-3}$$

So $$\frac{d}{dx}[h(x)] = \frac{d}{dx}(x^3 + 3x + 3x^{-1} + x^{-3})$$

$$= 3x^2 + 3 \times \frac{d}{dx}(x) + 3\frac{d}{dx}(x^{-1}) + (-3)x^{-4} \quad (☞)$$

$$= 3x^2 + 3 \times 1 + 3 \times (-1)x^{-2} - 3x^{-4}$$

$$= 3x^2 + 3 - \frac{3}{x^2} - \frac{3}{x^4} \quad \square$$

The derivative of kx^n was included in Table 1a, but as you can see you could have worked it out from the derivative of x^n and the constant multiple rule.

◆ Use the constant multiple rule to show that $[f(x) - g(x)]' = f'(x) - g'(x)$.

◇ We may certainly write, $f(x) - g(x) = f(x) + (-1)g(x)$,

so, using the sum rule we may write

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[(-1)g(x)]$$

and using the constant multiple rule we can write

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Thus, $[f(x) - g(x)]' = f'(x) - g'(x)$ $\square$

This last result shows that *the derivative of the difference of two functions is the difference of the derivatives*, i.e.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \qquad (10)$$

Question T10 Find the derivatives of the following functions:

(a) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^4$ (b) $\log_e \sqrt[3]{x}$

Remember, $\sqrt{x}$ represents the *positive* square root of x in this module.

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(c) $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$ (d) $(\pi-x)^2$

(e) $\sin(x+2)$ (f) $\left[\tan\left(x+\frac{\pi}{4}\right)\right](1-\tan x)$

$\left[\textit{Hint: } \tan(A+B)=\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)\right]$ □

Question T11 If k is a constant and $f(x)$ a function, write down the definition of $f'(x)$ and $[kf(x)]'$. What assumption must you make about limits to justify the conclusion $[kf(x)]'=kf'(x)$? □

3.4 Derivative of a product of functions

This subsection introduces yet another way of combining functions and obtaining the derivative of the result. This is the *product* of two functions, $f(x)g(x)$. Our aim is to express the derivative of such a product in terms of the derivatives of the functions $f(x)$ and $g(x)$ that are multiplied together to form the product, but the answer is not as obvious as it was for the sum or difference of functions. The 'obvious' answer that the derivative of a product is the product of the derivatives is *wrong*.

You cannot simply multiply derivatives.

To convince yourself of this consider $f(x)=x^2$ and $g(x)=x^3$. Then $f(x)g(x)=x^2x^3=x^5$. We can differentiate each of these functions using Table 1:

thus, $f(x)=x^2$ implies $f'(x)=2x$

$g(x)=x^3$ implies $g'(x)=3x^2$

and $f(x)g(x)=x^5$ implies $[f(x)g(x)]'=5x^4$

Clearly, $[f(x)g(x)]'\neq f'(x)\times g'(x)$

◆ Let $f(x)=2x$ and $g(x)=\frac{1}{4x}$

Find $f'(x)$, $g'(x)$, $[f(x)g(x)]'$ and show that $[f(x)g(x)]'\neq f'(x)\times g'(x)$.

◇ First we have $f'(x)=2$,

and $g(x)=\frac{1}{4x}=\frac{1}{4}x^{-1}$ so $g'(x)=-\frac{1}{4}x^{-2}$

Therefore $f'(x)\times g'(x)=2\times\left(-\frac{x^{-2}}{4}\right)=-\frac{1}{2}x^{-2}$

However, $f(x)g(x)=2x\times\frac{1}{4x}=\frac{1}{2}$

and the derivative of this is, $[f(x)g(x)]'=0$

So in this case $[f(x)g(x)]'\neq f'(x)\times g'(x)$. □

The correct determination of $[f(x)g(x)]'$ involves not only $f'(x)$ and $g'(x)$ *but also* $f(x)$ and $g(x)$. It is given by the **product rule** and is easier to express in symbols than in words:

product rule

product rule	$\frac{d}{dx}[f(x)g(x)]=g(x)\frac{df}{dx}+f(x)\frac{dg}{dx}$	(11)

This is an important rule and should be committed to memory.

The product rule can be derived from the basic definition of the derivative. We will, however, state the rule here without deriving it.

i.e. $[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$ or just $(fg)'=f'g+fg'$

The expression of this rule in terms of words is probably not very helpful:

The derivative of f times g equals the derivative of f times g plus f times the derivative of g.

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You may find Figure 4 a more useful memory aid.

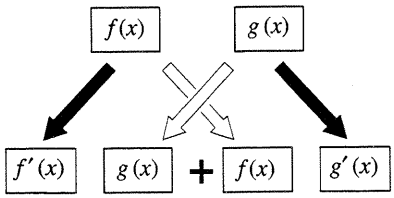


Figure 4 Differentiating a product of functions.

In the case of $f(x) = x^2$ and $g(x) = x^3$ (discussed earlier) where

$$f'(x) = 2x \quad \text{and} \quad g'(x) = 3x^2$$

the application of the product rule to $f(x)g(x)$ gives

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) = 2x \times x^3 + x^2 \times 3x^2 = 2x^4 + 3x^4 = 5x^4$$

and $5x^4$ is what we expect for the derivative of $f(x)g(x) = x^2x^3 = x^5$.

◆ If $f(x) = 2x$ and $g(x) = \dfrac{1}{4x}$ apply the product rule to find the derivative of $f(x)g(x)$.

◇ $f(x) = 2x$ and $g(x) = \dfrac{1}{4x}$

so $f'(x) = 2$ and $g'(x) = -\dfrac{1}{4x^2}$

thus $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

$$= 2 \times \frac{1}{4x} + 2x\left(-\frac{1}{4x^2}\right) = \frac{1}{2x} - \frac{1}{2x} = 0$$

which is correct since $f(x)g(x) = 2x\dfrac{1}{4x} = \dfrac{1}{2}$ is a constant and has derivative zero. □

Illustrations of the product rule

With the aid of the product rule we can now differentiate many more functions. The important step is to express the function that is to be differentiated in terms of the sum, difference or product of functions with known derivatives. Here are some examples.

Example 1 Differentiate $2 \sin x \cos x$.

Solution $2 \sin x \cos x$ may be written as a product of the functions

$$f(x) = 2 \sin x \quad \text{and} \quad g(x) = \cos x$$

for which $f'(x) = 2 \cos x$ and $g'(x) = -\sin x$

Using the product rule we can therefore write

$$\frac{d}{dx}(2 \sin x \cos x) = (2 \cos x)(\cos x) + (2 \sin x)(-\sin x) \quad (☞)$$

$$= 2(\cos^2 x - \sin^2 x) \quad □$$

Note that we have deliberately switched to Leibniz notation at this point.

Example 2 Differentiate $xe^x + \sec^2 x$.

Solution $xe^x + \sec^2 x$ is the sum of two functions xe^x and $\sec^2 x$, each of which is a product. So we differentiate xe^x first and then $\sec^2 x$ (using the product rule for each) and then add the answers using the sum rule.

First, xe^x may be written as the product of the functions

$$f(x) = x \quad \text{and} \quad g(x) = e^x$$

for which $f'(x) = 1$ and $g'(x) = e^x$

then, using the product rule we can write

$$\frac{d}{dx}(xe^x) = e^x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(e^x)$$

$$= e^x \times 1 + x \times e^x = (1 + x)e^x$$

Second, $\sec^2 x = \sec x \times \sec x$, therefore

$$f(x) = \sec x \quad \text{and} \quad g(x) = \sec x$$

giving us $f'(x) = \sec x \tan x$ and $g'(x) = \sec x \tan x$

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Then, again using the product rule we find

$$\frac{d}{dx}(\sec^2 x) = (\sec x \tan x) \times (\sec x) + (\sec x) \times (\sec x \tan x) = 2 \sec^2 x \tan x$$

Thus, combining these results, and using the sum rule we find

$$\frac{d}{dx}(xe^x + \sec^2 x) = \frac{d}{dx}(xe^x) + \frac{d}{dx}(\sec^2 x) = (1 + x)e^x + 2 \sec^2 x \tan x \quad \square$$

◆ The *kinetic energy* of a particle of mass m moving with a speed $v(t)$ that varies with time t, is $m[v(t)]^2/2$. Use the product rule to show that the rate of change of $m[v(t)]^2/2$ with time is $mv(t)\dfrac{dv}{dt}(t)$.

◇ Writing $[v(t)]^2 = v(t) \times v(t)$, and using the product rule, we obtain

$$\frac{d}{dt}[v(t)]^2 = v'(t)v(t) + v(t)v'(t) = 2v(t)v'(t)$$

Using the constant multiple rule, it follows that

$$\frac{d}{dt}\left\{\frac{1}{2}m[v(t)]^2\right\} = \frac{1}{2}m\frac{d}{dt}\{[v(t)]^2\}$$

$$= \frac{1}{2}m[2v(t)v'(t)] = mv(t)\frac{dv}{dt}(t) \quad \square$$

Question T12 Find the derivative of each of the following functions:
(a) $x^2 \log_e x$, (b) $(\sin x - 2\cos x)^2$, (c) $(1 - \sqrt{x})(1 + \sqrt[3]{x})$,
(d) e^{2x}, (e) $\sec x \cot x$, (f) $x^3F(x)$, where $F(x)$ is an arbitrary function. $\square$

Remember, $\sqrt{x}$ is the *positive* square root of x.

Question T13 The instantaneous motion of a particle moving along a straight line (call it the x-axis) can be described, at time t, in terms of its *displacement* $s_x(t)$ from a fixed reference position, its *velocity* $v_x(t)$ and its *acceleration* $a_x(t)$. (☞) These quantities are defined in such a way that

$$a_x(t) = \frac{dv_x}{dt}(t) \quad \text{and} \quad v_x(t) = \frac{ds_x}{dt}(t)$$

If the displacement is given as a function of time by

$$s_x(t) = [\cos(\omega t) - 2\sin(\omega t)]\exp(\alpha t) \quad \text{where } \omega \text{ and } \alpha \text{ are constants}$$

show that $a_x(t) = 2\alpha v_x(t) - (\omega^2 + \alpha^2)s_x(t)$ (☞) $\square$

Strictly speaking, displacement is a *vector* quantity that requires three *components* for its full specification in three dimensions. However, when dealing with one-dimensional motion it is conventional to refer to the single varying component (s_x in this case) as the displacement. Similar comments apply to velocity and acceleration.

You may find Question T13 quite challenging. It isn't difficult if you tackle it the right way, but the algebra will get rather fierce if you don't. Look at the answer if the going gets tough.

Question T14 This question extends the product rule to three functions. If $f(x)$, $g(x)$, $h(x)$ are any three functions, show that

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \quad \square$$

Question T15 Use the result developed in Question T14 to find

$$\frac{d}{dx}(e^x \log_e x \sin x) \quad \square$$

3.5 Derivative of a quotient of functions

It is frequently the case that we need to find the derivative of the quotient $\dfrac{f(x)}{g(x)}$ of two functions $f(x)$ and $g(x)$. (☞) The derivative of a quotient, like that of a product, depends not only on the values of $f'(x)$ and $g'(x)$, but also on $f(x)$ and $g(x)$. The **quotient rule** states:

We must of course ensure that $g(x) \neq 0$ if the quotient is to be well defined.

quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{[g(x)]^2} \qquad (12)$$

This is an important rule and should be committed to memory.

The quotient rule can be derived from the basic definition of the derivative. We will, however, state the rule here without deriving it.

i.e. $$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad \text{or just} \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

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You may find Figure 5 a useful memory aid.

reciprocal rule

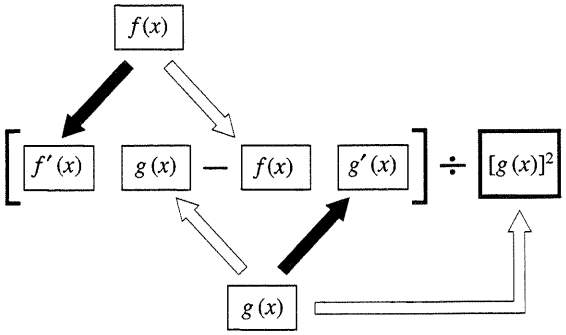


Figure 5 The quotient rule.

To apply this result to the function $y = \dfrac{\sin x}{x}$

take $f(x) = \sin x$ and $g(x) = x$

so that $f'(x) = \cos x$ and $g'(x) = 1$

Then $\dfrac{dy}{dx} = \dfrac{(\cos x)x - (\sin x) \times 1}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$ (☞)

The same result can be obtained by writing

$$y = \frac{\sin x}{x} = \sin x \times x^{-1}$$

then using the product rule

$$\frac{dy}{dx} = \cos x \times x^{-1} + \sin x \times (-x^{-2})$$

$$= \frac{x\cos x - \sin x}{x^2}$$

If you cannot remember the quotient rule it is always possible to use the product rule in this way.

◆ Write $y = x^3$ as x^5/x^2 and use the quotient rule to find dy/dx.

◇ Set $f(x) = x^5$ and $g(x) = x^2$

then $f'(x) = 5x^4$ and $g'(x) = 2x$

and $$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(5x^4)(x^2) - (x^5)(2x)}{(x^2)^2} = \frac{5x^6 - 2x^6}{x^4} = 3x^2$$

(and from Table 1 we know that the derivative of x^3 is $3x^2$, so we have verified that the quotient rule gives the correct answer in this particular case.) □

◆ Use the quotient rule to find $\dfrac{d}{dx}\left[\dfrac{1}{F(x)}\right]$ in terms of $F'(x)$ and $F(x)$.

◇ Putting $f(x) = 1$ and $g(x) = F(x)$ in the quotient formula, we obtain

the **reciprocal rule** $$\frac{d}{dx}\left[\frac{1}{F(x)}\right] = -\frac{F'(x)}{[F(x)]^2} \qquad (13)$$ □

You may find this useful, but there is no point in remembering it if you know the quotient rule since it is just a special case of that more general rule.

Question T16 Find dy/dx in each of the following cases:

(a) $y = \dfrac{x}{e^x}$ (b) $y = \dfrac{1 - x^2}{\cos x}$ (c) $y = \dfrac{x^5 - x^4 + x^3 - x^2}{(x+1)^2}$ (d) $y = \dfrac{x \log_e x}{\tan x}$ □

Question T17 If $f(x) = (x-1)^2$ and $g(x) = xe^x$, find the following:

(a) $\dfrac{d}{dx}[f(x) - g(x)]$ (b) $\dfrac{d}{dx}[f(x)g(x)]$ (c) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$ (d) $\dfrac{d}{dx}\left[\dfrac{g(x)}{f(x)}\right]$ □

In part (d) of Question T17, try using the reciprocal rule and the result from part (c).

The aim of the next question is to derive the quotient rule from the product rule. It begins by showing how to obtain the reciprocal rule *without* using the quotient rule.

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Question T18 (a) Let $f(x)$ be an arbitrary function and let $h(x) = 1/f(x)$ so that

$$f(x)h(x) = 1$$

Differentiate both sides of this equation and hence obtain a formula for $\frac{d}{dx}\left[\frac{1}{f(x)}\right]$ (similar to that given in Equation 13).

(b) Let $f(x)$ and $g(x)$ be any two functions, and write $\frac{f(x)}{g(x)} = f(x) \times \frac{1}{g(x)}$, then use the answer to part (a) and the product rule to derive the quotient rule. □

Question T19 Use $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, and the quotient rule to differentiate (a) $\tan x$, and (b) $\cot x$. □

Question T20 (a) Use the reciprocal rule to find $\frac{d}{dx}(e^{-x})$.

(b) The function $f(x) = \frac{1}{2}(e^x + e^{-x})$ is sometimes known as the *hyperbolic cosine* of x (and is denoted by $\cosh x$).

Find $\frac{df}{dx}$ and show that $[f(x)]^2 - \left(\frac{df}{dx}\right)^2 = 1$. □

Question T21 Let $F(x) = G\frac{Mm}{(x-a)^2}$ where G, M, m and a are constants. Find $F'(x)$. □

Question T22 Using $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$, along with the reciprocal rule, find the derivatives of (a) $\sec x$ and (b) $\operatorname{cosec} x$. □

4 More about logarithmic and exponential functions

4.1 Why exp(x) is considered to be special

Of all the functions listed in Table 1, $f(x) = \exp(x) = e^x$ is the only one which is its own derivative. Far from being a technical curiosity this fact turns out to be of great significance and ultimately explains why the exponential function plays such an important role in physics. (☞) It also has an interesting interpretation in terms of the graph of $y = e^x$, shown in Figure 6. Since $f'(x) = e^x$ the gradient of the tangent to the graph at the point A which has coordinates $x = a$ and $y = e^a$ is $f'(a) = e^a$ which is the same as the height of A above C.

Exponential functions describe the decay of radioactive samples, the growth of unrestricted populations, the dying oscillations of an appropriately damped pendulum and many other physically important phenomena.

So the rate of change of $f(x)$ when $x = a$ is simply $f(a)$, that is, the rate of change of the exponential function at any point is equal to the value of the function itself at that point. In Figure 6 the point B has coordinates $(a - 1, 0)$. Since A is the point (a, e^a), the line AB has a gradient of

$$\frac{CA}{BC} = \frac{e^a}{1} = e^a$$

and is therefore the tangent to the graph at A. So, to draw a tangent to $y = e^x$ at (a, e^a), simply join (a, e^a) to $(a - 1, 0)$ and the resulting line is bound to have a gradient of e^a.

◆ Is $f(x) = e^x$ the only function which is its own derivative? (Think about the rules of differentiation that you already know.)

◇ No. The derivative of a constant multiple of a function is the constant multiple times the derivative, so we have

$$\frac{d}{dx}(Ae^x) = Ae^x \qquad \text{for any constant } A$$

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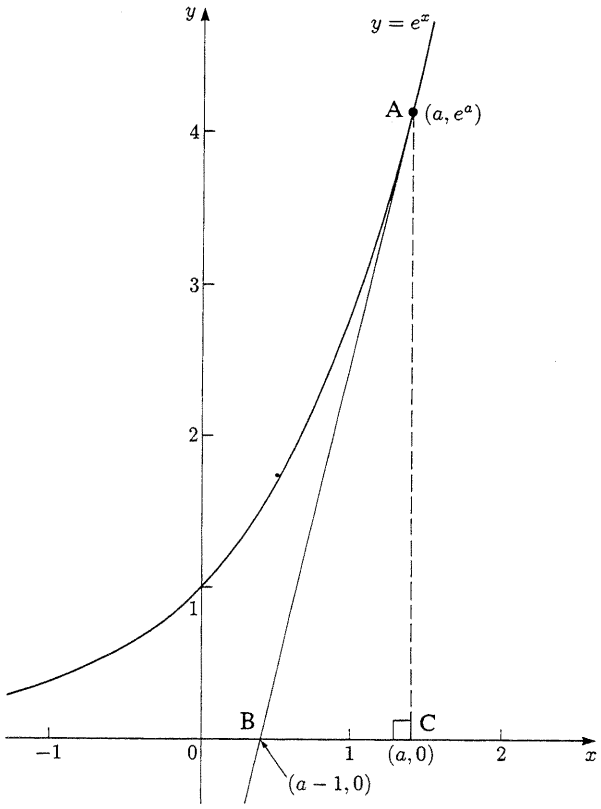


Figure 6 The graph of $y = e^x$.

Thus, any constant multiple of the exponential function, such as Ae^x, is its own derivative. However, apart from functions of this kind there are no other functions that are their own derivatives. (☞) □

The function $f(x) = 0$ is equal to its own derivative (0) at every point, but this is a special case of Ae^x corresponding to $A = 0$.

◆ Using the product rule find the derivatives of (a) e^{2x}, (b) e^{3x}, (c) e^{4x}.

Deduce the formula for the derivative of e^{nx} where n is any positive integer.

◇ (a) Since $e^{2x} = e^x \times e^x$ the product rule gives $\dfrac{d}{dx}(e^{2x}) = 2e^{2x}$

(b) $e^{3x} = e^x \times e^{2x}$, so $\dfrac{d}{dx}(e^{3x}) = e^x \times 2e^{2x} + e^x e^{2x} = 3e^{3x}$

(c) $e^{4x} = e^{2x} \times e^{2x} = e^{2x} \times 2e^{2x} + 2e^{2x} e^{2x} = 4e^{4x}$

Therefore, generally, if n is any positive integer $\dfrac{d}{dx}(e^{nx}) = ne^{nx}$. □

A similar result holds true even when n is not a positive integer. This was given in Table 1, but it deserves to be emphasized again here.

$$\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{for any constant } k \tag{14}$$

Question T23 If $y = \dfrac{Ce^x}{1 + Ce^x}$ for some constant C, show that $\dfrac{dy}{dx} = y(1 - y)$. □

Question T24 Differentiate $f(x) = (e^x + e^{-x})(e^x - e^{-x})$. □

4.2 Derivative of a^x

The previous subsection considered the derivative of e^x. However, functions such as 2^x and π^x sometimes arise in physical applications, and so it is necessary to know how to differentiate any function of the form a^x, where the positive number a being raised to the power x is not necessarily e. As an example we will consider the differentiation of 2^x, but the method is of general applicability. First note that exp is the *inverse function* of $\log_e$, this means that exp reverses the effect of $\log_e$ so that

$$e^{\log_e 2} = \exp(\log_e 2) = 2$$

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since $2 = \exp(\log_e 2)$ $\quad 2^x = [\exp(\log_e 2)]^x = \exp(x \log_e 2)$

So, $\dfrac{d}{dx}(2^x) = \dfrac{d}{dx}[\exp(x \log_e 2)]$

Now, from Equation 14 (or Table 1) we know that

$$\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{for any constant } k$$

and, if we choose k to be $\log_e 2$, this gives us

$$\frac{d}{dx}(2^x) = \frac{d}{dx}[\exp(x \log_e 2)] = (\log_e 2)[\exp(x \log_e 2)] = (\log_e 2)2^x$$

A similar argument applies with any positive number, a say, in place of 2. So we have the result:

$$\frac{d}{dx}(a^x) = (\log_e a)a^x \quad (a > 0) \tag{15}$$

Rather than trying to remember this result you would probably be wiser to try to remember the method that was used to obtain it. Notice the way that the properties of exponentials and logarithms have been used to express a function of the form a^x in terms of e^{kx}, and how the simple properties of e^{kx} have then been exploited.

◆ Find $\dfrac{d}{dx}(\pi^x)$.

◇ Here $a = \pi$ so the derivative is $(\log_e \pi)\pi^x$. □

◆ Apply the formula for differentiating a^x (Equation 14) when $a = e$.

◇ The result is $(\log_e a)a^x$ and if $a = e$ then, since $\log_e e = 1$, we have $\dfrac{d}{dx}(e^x) = e^x$ as before. □

Question T25 Differentiate each of the following functions:

(a) $f(x) = 3^x$, (b) $f(x) = 3^{x-1}$, (c) $f(x) = (2+3)^x$, (d) $f(x) = 2^x e^x$. □

4.3 Derivative of $\log_a x$

As stated in Table 1:

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

From this it is easy to deduce the derivative of $\log_e kx$, where k is a constant

$$\log_e(kx) = \log_e k + \log_e x$$

so $\dfrac{d}{dx}(\log_e(kx)) = \dfrac{d}{dx}(\log_e k) + \dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

Note that k plays no part in the derivative because k is constant, so $\dfrac{d}{dx}(\log_e k) = 0$.

If we had been compelled to differentiate $\log_{10} x$ or $\log_2 x$ the problem would have been a little more difficult, but still not intractable. The general method for differentiating logs to an arbitrary base, such as $\log_a x$ is remarkably similar to that for dealing with a^x: it involves expressing $\log_a x$ in terms of $\log_e x$ which can be differentiated easily. As an example, let us consider $\log_{10} x$. To express it in terms of $\log_e x$ we write

$$y = \log_{10} x$$

then $\quad x = 10^y$

and, taking logarithms to the base e of both sides of this equation, we obtain

$$\log_e x = \log_e(10^y) = y \log_e 10$$

and recalling that $y = \log_{10} x$ this can be rearranged to give

$$y = \log_{10} x = \frac{\log_e x}{\log_e 10}$$

In more general terms

$$\log_c b = \frac{\log_a b}{\log_a c}$$

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We may now easily differentiate $\log_{10} x$ because $1/\log_e 10$ is a constant, so

$$\frac{d}{dx}(\log_{10} x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e 10}\right)$$

$$= \left(\frac{1}{\log_e 10}\right)\frac{d}{dx}(\log_e x) = \left(\frac{1}{\log_e 10}\right) \times \frac{1}{x} = \frac{1}{x \log_e 10}$$

A similar argument applies for any positive base a giving the general result:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} \qquad (16)$$

Once again, you should try to remember the general method rather than just the result.

◆ If a = e, check that the above formula gives our previous result for the derivative of $\log_e x$.

◇ $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$. If a = e then $\log_e e = 1$, and the result is $1/x$ as expected. □

◆ Differentiate $\log_\pi x$.

◇ From Equation16:

$$\frac{d}{dx}(\log_\pi x) = \frac{1}{x \log_e \pi} \quad \square$$

Question T26 Differentiate each of the following functions:

(a) $f(x) = \log_2 x$

(b) $f(x) = \log_{10} x^2$

(c) $f(x) = \dfrac{\log_2 x}{\log_{10} x}$ [*Hint*: $\log_a b = (\log_a c)(\log_c b)$]

(d) $f(x) = a^x \log_a x$ where a is a positive constant. □

4.4 Resisted motion under gravity: an example

Imagine a parachutist falling to Earth with velocity, $v_x(t)$ downwards, at time t. For convenience we will choose the positive x-axis to be the downward vertical, with the origin at the point where the parachute opens. The parachutist is subject to two forces, gravity acting downwards and the resistance of the air on his parachute acting upwards. Suppose that the resistive force is proportional to $[v_x(t)]^2$ with a constant of proportionality k (which has to be determined experimentally). It follows from *Newton's second law* that the equation governing the motion is

$$m\frac{d}{dt}[v_x(t)] = mg - k[v_x(t)]^2 \qquad (17)$$

where m is the mass of the parachutist and g is the magnitude of the acceleration due to gravity. The forces on the right-hand side have opposite sign because they act in opposite directions.

Now, suppose you want to check that the following expression for $v_x(t)$ satisfies Equation 17

$$v_x(t) = \sqrt{\frac{mg}{k}}\left[\frac{1 + B\exp(-2t\sqrt{gk/m})}{1 - B\exp(-2t\sqrt{gk/m})}\right] \qquad (18)$$

where B is a constant (determined by the speed of the parachutist at time $t = 0$). It appears that we must first differentiate $v_x(t)$, which is a rather complicated multiple of the quotient of two functions.

We could certainly proceed directly, and just differentiate the function as it stands, but a little thought should tell you that the algebra is going to get very nasty; so it is worth trying to simplify the expression. To simplify the notation let $A = \sqrt{mg/k}$ and $\alpha = 2\sqrt{gk/m}$, and let us write v_x rather than $v_x(t)$. So we have a simplified version of Equation 18

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terminal velocity

$$v_x = A\left[\frac{1+B\exp(-\alpha t)}{1-B\exp(-\alpha t)}\right] \tag{19}$$

Now, using the quotient rule to differentiate this equation with respect to t, we find

$$\frac{dv_x}{dt} = A\left\{\frac{\left[\frac{d}{dt}(1+Be^{-\alpha t})\right](1-Be^{-\alpha t}) - \left[\frac{d}{dt}(1-Be^{-\alpha t})\right](1+Be^{-\alpha t})}{(1-Be^{-\alpha t})^2}\right\}$$

$$\frac{dv_x}{dt} = A\left[\frac{(-B\alpha e^{-\alpha t})(1-Be^{-\alpha t}) - (B\alpha e^{-\alpha t})(1+Be^{-\alpha t})}{(1-Be^{-\alpha t})^2}\right]$$

$$\frac{dv_x}{dt} = A\left[\frac{(-B\alpha e^{-\alpha t} + B^2\alpha e^{-2\alpha t}) - (B\alpha e^{-\alpha t} + B^2\alpha e^{-2\alpha t})}{(1-Be^{-\alpha t})^2}\right]$$

$$\frac{dv_x}{dt} = A\left[\frac{-2B\alpha e^{-\alpha t}}{(1-Be^{-\alpha t})^2}\right]$$

Substituting this expression for $dv_x(t)/dt$ into the left-hand side of Equation 17 and using the fact that $A\alpha = 2g$ we obtain

$$mA\left[\frac{-2B\alpha e^{-\alpha t}}{(1-Be^{-\alpha t})^2}\right] = \frac{-4mgBe^{-\alpha t}}{(1-Be^{-\alpha t})^2}$$

and substituting the same expression for $dv_x(t)/dt$ into the right-hand side of Equation 17 and using the fact that $kA^2 = mg$ we obtain

$$mg - mg\left(\frac{1+Be^{-\alpha t}}{1-Be^{-\alpha t}}\right)^2 = mg\left[\frac{(1-Be^{-\alpha t})^2 - (1+Be^{-\alpha t})^2}{(1-Be^{-\alpha t})^2}\right]$$

$$= \frac{-4mgBe^{-\alpha t}}{(1-Be^{-\alpha t})^2}$$

Thus, the two sides of Equation 17 are indeed equal if v_x has the form given in Equation 18.

This really completes the differentiation and subsequent manipulation, but having introduced an expression (Equation 18) for the downward velocity under gravity in the presence of a resistive force it is worth noting at least one of its mathematical properties.

◆ What is the behaviour of $v_x(t)$ as t becomes large? In other words, what is $\lim_{t\to\infty}[v_x(t)]$?

◇ As t becomes large $Be^{-\alpha t}$ becomes small, and the velocity approaches the value $\sqrt{\frac{mg}{k}}\frac{1+0}{1-0} = \sqrt{\frac{mg}{k}}$. This final constant value of the velocity is called the **terminal velocity**. Its existence is a striking feature of resisted motion. □

◆ What is the terminal velocity if $m = 80$ kg, $k = 30$ kg m^{-1} and $g = 9.81$ m s^{-2}?

◇ Terminal velocity $= \sqrt{\frac{mg}{k}} = \sqrt{\frac{(80\,\text{kg})\times(9.81\,\text{m s}^{-2})}{(30\,\text{kg m}^{-1})}} \approx 5.1\,\text{m s}^{-1}$. □

Question T27 Show that $y(x) = \dfrac{\log_e 2 - x}{\log_e 2 + x}$ satisfies the equation

$$2y'(x)\log_e 2 + [1+y(x)]^2$$ □

5 Closing items

5.1 Module summary

1 A function, f say, is a rule that assigns a single value of the dependent variable, y say, (in the codomain) to each value of the independent variable x in the domain of the function, such that $y = f(x)$.

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2 The *rate of change*, or *derivative* of a function $f(x)$ at $x = a$, can be interpreted as the *gradient* of the graph of the function at the point $x = a$. Some standard derivatives are given in Table 1a (repeated here for reference).

3 The derivative is defined more formally in terms of a *limit*, and may be represented in a variety of ways. If $y = f(x)$

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

4 The *sum*, *constant multiple*, *product* and *quotient* rules for differentiating combinations of functions are as follows:

sum rule $(f+g)' = f' + g'$

constant multiple rule $(kf)' = kf'$

product rule $(fg)' = f'g + fg'$

quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

5 The derivatives of the standard functions are given in Table 1.

6 The derivatives of a^x and $\log_a x$ are given by

$$\frac{d}{dx}(a^x) = (\log_e a)a^x \qquad \text{(Eqn 15)}$$

and $$\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} \qquad \text{(Eqn 16)}$$

7 You should be aware that logarithms to different bases are related by

$$\log_a b = (\log_a c)(\log_c b)$$

Table 1(a) Some standard derivatives.

$f(x)$	$f'(x)$
k (constant)	0
kx^n	nkx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx \cot kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\exp(kx)$	$k\exp(kx)$
$\log_e(kx)$	$1/x$

5.2 Achievements

Having completed the module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Define the derivative of a function (at a point) as a limit and give the definition a graphical interpretation.

A3 Use the definition of the derivative as a limit to calculate the derivative of simple functions.

A4 Know the derivatives of a range of standard functions.

A5 Use addition, subtraction, multiplication and division to express a given function in terms of 'simpler' ones in appropriate cases.

A6 Differentiate a constant multiple of a function.

A7 Apply the rule for differentiating the sum (and difference) of functions.

A8 Apply the product rule of differentiation.

A9 Apply the quotient rule of differentiation.

A10 Use the properties of logarithms and exponentials to determine the derivatives of $\log_a x$ and a^x.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A1*, *A2* and *A4*) Write down the derivative of $f(x) = \log_e x$. As x increases from zero how does the gradient of the tangent to the graph change? What is the behaviour of $\log_e x$ as x approaches infinity?

Question E2 (*A3*) Using the definition of the derivative as a limit, differentiate $f(x) = 1/x^2$.

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Question E3 (*A4*, *A5* and *A6*) Find dy/dx for each of the following functions:

(a) $y = 2x(x-1)(x+2)$

(b) $y = \sin(x + \pi/4)$ (*Hint*: $\sin(A+B) = \sin A \cos B + \cos A \sin B$)

(c) $y = \log_e x^2$

Question E4 (*A4* to *A9*) Find $f'(x)$ for each of the following functions:

(a) $f(x) = x\cos x + x^2 \sin x$

(b) $f(x) = \left(x - \dfrac{1}{x}\right)^2 e^x$

(c) $f(x) = \dfrac{1 + x + x^2}{1 - x^2}$

(d) $f(x) = \dfrac{\sqrt{x}\sin x}{\log_e x}$

(e) $f(x) = \dfrac{\exp(x + \log_e x)}{\sqrt{x}}$

(f) $f(x) = a^x x^a$

Question E5 (*A4*, *A5*, *A6*, *A8* and *A9*) If n is a positive integer, the series

$$1 + x + x^2 + x^3 + \ldots + x^n \quad \text{has the sum} \quad \frac{x^{n+1} - 1}{x - 1}$$

show that the sum of the series

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} \quad \text{is} \quad \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

Think about this question, the solution is easy if you handle it in the right way.

Question E6 (*A7* and *A8*) If f and g are functions show that

$$\left(\frac{1}{f(x)g(x)}\right)' = \left(\frac{1}{f(x)}\right)' \frac{1}{g(x)} + \frac{1}{f(x)}\left(\frac{1}{g(x)}\right)'$$

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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6 Answers and comments

Fast track answers

F1 (a) $f'(x) = x^2 \cos x + 2x \sin x + (1/x)$.

(b) $f'(x) = (x^3 - 6x)(4x + 5) + (3x^2 - 6)(2x^2 + 5x - 1)$
$= 10x^4 + 20x^3 - 39x^2 - 60x + 6.$

(c) $f(x) = 1/\sin x = \text{cosec } x$, so $f'(x) = -\text{cosec } x \cot x$.

(d) $f'(x) = \dfrac{(2x+4)(x^2+9) - (x^2+4x-1)2x}{(x^2+9)^2}$

(e) $f'(x) = \dfrac{(x\sec^2 x + \tan x)(3x^2 - 2x + 1) - x\tan x(6x-2)}{(3x^2-2x+1)^2}$

(f) $f'(x) = (\log_e 2)\, 2^x + 2x$ Note that e^2 is a constant.

(g) $f(x) = (e^x + e^{-x})(e^x + e^{-x})$, so using the product rule we find

$f'(x) = (e^x + e^{-x})(e^x - e^{-x}) + (e^x - e^{-x})(e^x + e^{-x})$
$= 2(e^{2x} - e^{-2x})$

F2 $s_x(t) = \exp(-\alpha t)[A\cos(\omega t) + B\sin(\omega t)]$

$$v_x(t) = \frac{d}{dt}s_x(t) = \exp(-\alpha t)[-A\omega\sin(\omega t) + B\omega\cos(\omega t)] - \alpha\exp(-\alpha t)[A\cos(\omega t) + B\sin(\omega t)]$$

So $v_x(t) = \omega\exp(-\alpha t)[-A\sin(\omega t) + B\cos(\omega t)] - \alpha s_x(t)$

$a_x(t) = v_x'(t)$
$= \omega^2\exp(-\alpha t)[-A\cos(\omega t) - B\sin(\omega t)]$
$- \alpha\omega\exp(-\alpha t)[-A\sin(\omega t) + B\cos(\omega t)] - \alpha v_x(t)$

So $a_x(t) = -\omega^2 s_x(t) - \alpha\omega\exp(-\alpha t)[-A\sin(\omega t) + B\cos(\omega t)] - \alpha v_x(t)$
$= -\omega^2 s_x(t) - \alpha v_x(t) - \alpha^2 s_x(t) - \alpha v_x(t)$

using the expression for $v_x(t)$, and therefore

$a_x(t) + 2\alpha v_x(t) + (\omega^2 + \alpha^2)s_x(t) = 0$

Ready to study answers

R1 (a) $\log_{10} xy$ (b) $(2\times 2)^3 = 64$ (c) $\log_e(x/y)$ (d) e^{x-y} (e) e^{2x-2y} (f) $3\log_e x = \log_e x^3$.

Consult *logarithm* and *exponential* in the *Glossary* for further details.

R2 (a) $[f(x) + g(x)]^2 = (\cos x + \sin x)^2$
$= \cos^2 x + 2\cos x \sin x + \sin^2 x$

Since $\sin^2 x + \cos^2 x = 1$ and $\sin(2x) = 2\sin x\cos x$, for all x, we can write $[f(x) + g(x)]^2 = 1 + \sin(2x)$.

(b) $f(x)h(x) + g(x)/h(x) = \cos x\tan x + \sin x/\tan x$
$= \cos x \times \dfrac{\sin x}{\cos x} + \sin x \times \dfrac{\cos x}{\sin x} = \sin x + \cos x$

(c) $\dfrac{f(x) - g(x)}{1 + h(x)} = \dfrac{\cos x - \sin x}{1 + \tan x}$

which cannot be simplified.

Consult *trigonometric functions* and *trigonometric identities* in the *Glossary* for further details.

R3 $f(1) = -0.75$, $f(0) = -1$, $f(-2) = 0$, $|f(0)| = 1$, $|f(0) - f(-2)| = 1$.

The graph of $y = \dfrac{x^2}{4} - 1$ is shown in Figure 7.

Consult *functions* and *modulus* in the *Glossary* for further details.

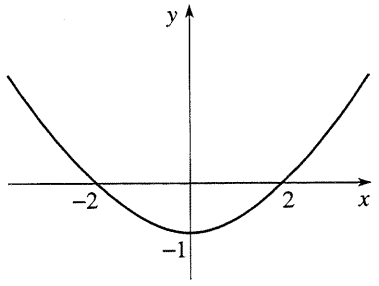


Figure 7 See Answer R3.

R4 (a) $2\log_{10} x$, (b) 10^x, (c) 2.

Consult *inverse functions* in the *Glossary* for further details.

Text answers

T1 (a) $f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$, and $g(x/2) = 2(x/2) = x$.

(b) We need $x^2 + 1 = 2x$, i.e. $x^2 - 2x + 1 = 0$. The solution of this is $x = 1$.

(c) $f(x+0.1) - f(x) = (x+0.1)^2 + 1 - (x^2+1)$
$= x^2 + 0.2x + 0.01 + 1 - x^2 - 1 = 0.2x + 0.01$

So, $\dfrac{f(x+0.1) - f(x)}{0.1} = \dfrac{0.2x + 0.01}{0.1} = 2x + 0.1$

If this equals 0.2 we must have $2x = 0.1$, which gives $x = 0.05$.

T2 (a) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{h}\left[\dfrac{1}{\omega(x+h)} - \dfrac{1}{\omega x}\right]$

$= \dfrac{1}{\omega} \times \dfrac{1}{h}\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)$

which is $1/\omega$ times the expression discussed earlier in this subsection, and so

$$f'(x) = \frac{1}{\omega}\left(-\frac{1}{x^2}\right) = -\frac{1}{\omega x^2}$$

(b) g is the same function as f and therefore

$$g'(t) = -\frac{1}{\omega t^2} \text{ and } g'(2t) = -\frac{1}{\omega(2t)^2} = -\frac{1}{4\omega t^2}$$

T3 (a) Trying various small values (either positive or negative) for h, you should have convinced yourself that

$\lim_{h\to 0}\left[\dfrac{\exp(h) - 1}{h}\right] = 1$, for example

$$\frac{\exp(0.001) - \exp(0)}{0.001} \approx \frac{1.001\,000\,500\,17 - 1}{0.001}$$
$$\approx 1.000\,500\,17$$

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(b) Part (a) should have convinced you that

$$f'(0) = \lim_{h\to 0}\left[\frac{\exp(h)-\exp(0)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{\exp(h)-1}{h}\right] = 1$$

(c) Keeping in mind the result of part (b)

$$f'(x) = \lim_{h\to 0}\left[\frac{\exp(x+h)-\exp(x)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{\exp(x)\exp(h)-\exp(x)}{h}\right]$$
$$= \lim_{h\to 0}\left[\exp(x)\frac{\exp(h)-1}{h}\right]$$
$$= \exp(x)\lim_{h\to 0}\left[\frac{\exp(h)-1}{h}\right] = \exp(x)$$

T4 If $f(x) = x^n$ then, using the binomial theorem

$$f(x+h) = (x+h)^n$$
$$= x^n + nx^{n-1}h + \tfrac{1}{2}n(n-1)x^{n-2}h^2 + \ldots + h^n$$

So, $$\frac{f(x+h)-f(x)}{h} = nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}h + \ldots + h^{n-1}$$

All the terms except the first involve a positive power of h, so the limit of each of them is zero, giving the limit of right-hand side as nx^{n-1} as required.

(This result holds true even when n is not a positive integer, but the above demonstration is not sufficiently general to prove that.)

T5 The gradient the graph of $y = \sin x$ at any value of x, is given by $dy/dx = \cos x$ (from Table 1).

(a) To find where the gradient is 0 we need to find the values of x for which $\cos x = 0$, these are

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \text{etc.}$$

(b) In this case we need to solve $\cos x = -1$, which gives $x = \pm\pi, \pm 3\pi, \pm 5\pi$, etc.

(c) In this case we need to solve $\cos x = +1$, which gives $x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi$, etc.

T6 (a) Table 2 shows approximate values of $\dfrac{\cos(h)-1}{h}$ and $\dfrac{\sin h}{h}$ for various values of h.

Table 2 See Answer T6.

h	$\dfrac{\cos(h)-1}{h}$	$\dfrac{\sin h}{h}$
0.1	−0.05	0.998
−0.1	0.05	0.998
0.01	−0.005	0.999 98
−0.01	0.005	0.999 98
0.0001	−0.000 05	0.999 999 998
−0.0001	0.000 05	0.999 999 998

Thus it seems very likely that

$$\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] = 0 \text{ and } \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$$

(These results are true, but we will not prove them rigorously.)

(b) $f(x) = \sin x$, so $f(x+h) = \sin(x+h)$

and (using the hint)

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}$$
$$= \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$
$$= \sin x\frac{\cos(h)-1}{h} + \cos x\frac{\sin h}{h}$$

Since $\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] = 0$ and $\lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$

we obtain

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = (\sin x)\times 0 + (\cos x)\times 1$$
$$= \cos x$$

In other words $\dfrac{d}{dx}(\sin x) = \cos x$.

T7 (a) The derivative of $\tan x$ is $\sec^2 x$ which is always positive because it is a square. Since e^x is always positive and is its own derivative, e^x always has a positive derivative.

The last function with a positive derivative is $\log_e x$. This involves a tricky point. Since $f(x) = \log_e x$ it is only defined for $x > 0$ so is its derivative, $f'(x)$. In its domain $f'(x)$ takes the same value as $1/x$, and on this domain $1/x$ is positive. (Many authors prefer to discuss the function $\log_e |x|$ since this function has $1/x$ as its derivative for all values of $x \neq 0$.)

(b) Since the derivative of $\cot x$ is $-\text{cosec}^2 x$ it is always negative.

(c) All the other derivatives in Table 1b can take both positive and negative values.

T8 (a) $y = 2 + \sqrt[3]{x} + e^x = 2 + x^{1/3} + e^x$ is the sum of three functions each belonging to Table 1.

So,

$$\frac{dy}{dx} = \frac{d}{dx}(2 + x^{1/3} + e^x) = \frac{d}{dx}(2) + \frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(e^x)$$
$$= 0 + \frac{1}{3}x^{-2/3} + e^x = \frac{1}{3}x^{-2/3} + e^x$$

(b) Expand $(1+\sqrt{x})^2$ to find

$$1 + 2\sqrt{x} + (\sqrt{x})^2 = 1 + \sqrt{x} + \sqrt{x} + x$$

(We have written $2\sqrt{x}$ as $\sqrt{x} + \sqrt{x}$ in order to express it as a sum of functions appearing in Table 1. In Subsection 3.3 another rule is developed which makes this unnecessary.) Hence

$$\frac{dy}{dx} = \frac{d}{dx}\left(1+\sqrt{x}\right)^2 = \frac{d}{dx}\left(1+\sqrt{x}+\sqrt{x}+x\right)$$
$$= 0 + \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-1/2} + 1 = \frac{1}{\sqrt{x}} + 1$$

(c) $\log_e(xe^x) = \log_e x + \log_e(e^x) = \log_e x + x$

Hence

$$\frac{dy}{dx} = \frac{d}{dx}(\log_e xe^x) = \frac{d}{dx}(\log_e x + x) = \frac{1}{x} + 1$$

(d) $$\cot x\sin x + \tan x\cos x = \frac{\cos x}{\sin x}\times\sin x + \frac{\sin x}{\cos x}\times\cos x$$
$$= \cos x + \sin x$$

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Hence

$$\frac{dy}{dx} = \frac{d}{dx}(\cot x \sin x + \tan x \cos x) = \frac{d}{dx}(\cos x + \sin x)$$
$$= -\sin x + \cos x$$

T9 You would have to assume (correctly, though we will not prove it) that the limit of a sum equals the sum of the limits.

T10 (a) Expanding the bracket gives us

$$x^2 - 4x + 6 - 4/x + 1/x^2 = x^2 - 4x + 6 - 4x^{-1} + x^{-2}$$

So, $\frac{d}{dx}\left[\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^4\right] = 2x - 4 - 4(-1)x^{-2} + (-2)x^{-3}$

$$= 2x - 4 + 4/x^2 - 2/x^3$$

(b) $\log_e \sqrt[3]{x} = \log_e x^{1/3} = \frac{1}{3}\log_e x$

So $\frac{d}{dx}\log_e \sqrt[3]{x} = \frac{d}{dx}\left(\frac{1}{3}\log_e x\right) = \frac{1}{3}\frac{d}{dx}\log_e x = \frac{1}{3x}$

(c) $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = \left(\sin\frac{x}{2}\right)^2 + 2\sin\frac{x}{2}\cos\frac{x}{2} + \left(\cos\frac{x}{2}\right)^2$

Since $\sin^2 A + \cos^2 A = 1$, and $2 \sin A \cos A = \sin(2A)$ for any angle A, this becomes $1 + \sin x$ with the derivative

$$\frac{d}{dx}(1 + \sin x) = 0 + \cos x = \cos x$$

(d) Expanding the bracket gives us

$$(\pi - x)^2 = \pi^2 - 2\pi x + x^2$$

Now π^2 is just a constant and so has a derivative of 0.

Therefore $\frac{d}{dx}[(\pi - x)^2] = 0 - 2\pi + 2x = -2(\pi - x)$

(e) $\sin(x + 2)$ does not appear in Table 1 so we need to expand it using $\sin(A + B) = \sin A \cos B + \cos A \sin B$ to find

$$\sin(x + 2) = \sin x \cos 2 + \cos x \sin 2$$

Note that cos 2 and sin 2 are just constants, so the derivative is

$$\frac{d}{dx}[\sin(x + 2)] = \cos 2 \frac{d}{dx}(\sin x) + \sin 2 \frac{d}{dx}(\cos x)$$
$$= \cos 2 \cos x - \sin 2 \sin x = \cos(x + 2)$$

(since $\cos(A + B) = \cos A \cos B - \sin A \sin B$)

(f) Using $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ the function can be written as

$$\frac{\tan x + \tan(\pi/4)}{1 - \tan x \tan(\pi/4)}(1 - \tan x)$$

We can use $\tan(\pi/4) = 1$ to simply this expression to $\tan x + \tan(\pi/4) = 1 + \tan x$ which has the derivative $\sec^2 x$.

T11 By definition

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$[kf(x)]' = \lim_{h\to 0}\left[\frac{kf(x+h) - kf(x)}{h}\right]$$

Therefore we can write

$$[kf(x)]' = \lim_{h\to 0} k\left[\frac{f(x+h) - f(x)}{h}\right]$$
$$= k\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$
$$= kf'(x)$$

We have assumed (correctly) that the limit of a constant multiple of an expression is the same as the constant multiple of the limit of the expression.

T12 (a) If $f(x) = x^2$ and $g(x) = \log_e x$, then

$f(x)g(x) = x^2 \log_e x$

$f'(x) = 2x$ and $g'(x) = 1/x$

So, $\frac{d}{dx}(x^2 \log_e x) = 2x \log_e x + x^2 \times \frac{1}{x} = x(2 \log_e x + 1)$

(b) Frequently there is more than one way to break down the given function in terms of simpler functions the derivatives of which we know. In this exercise we could multiply out the brackets, but that would still involve using the product rule, so it is shorter to take

$f(x) = (\sin x - 2 \cos x)$ and $g(x) = (\sin x - 2 \cos x)$

so that the given function is $f(x)g(x)$, then

$f'(x) = (\cos x + 2 \sin x)$ and $g'(x) = (\cos x + 2 \sin x)$

The derivative of the function $f(x)g(x)$ is therefore

$$(\cos x + 2 \sin x)(\sin x - 2 \cos x) + (\sin x - 2 \cos x)$$
$$\times (\cos x + 2 \sin x) = 2(\cos x + 2 \sin x)(\sin x - 2 \cos x)$$

(This sort of problem is better treated by another method known as the *chain rule*. This rule is discussed elsewhere in the differentiation block of *FLAP*.)

(c) This is probably a case where it is easier to expand the brackets and then differentiate, rather than to use the product rule directly. However, to use the product rule directly, put

$$f(x) = 1 - \sqrt{x} = 1 - x^{1/2} \text{ and } g(x) = 1 + \sqrt[3]{x} = 1 + x^{1/3}$$

Therefore $f'(x) = -\frac{1}{2}x^{-1/2}$ and $g'(x) = \frac{1}{3}x^{-2/3}$

Therefore the derivative is

$$(-\tfrac{1}{2}x^{-1/2})(1 + x^{1/3}) + (1 - x^{1/2})\tfrac{1}{3}x^{-2/3}$$
$$= -\tfrac{1}{2}x^{-1/2} - \tfrac{5}{6}x^{-1/6} + \tfrac{1}{3}x^{-2/3}$$

(d) Take $f(x) = e^x$ and $g(x) = e^x$ then $f'(x) = e^x$ and $g'(x) = e^x$

So, $\frac{d}{dx}(e^{2x}) = e^x e^x + e^x e^x = 2e^{2x}$

as can be verified from Table 1.

(e) Using the product rule (and without thinking!) take $f(x) = \sec x$ and $g(x) = \cot x$

$f'(x) = \sec x \tan x$ and $g'(x) = -\text{cosec}^2 x$

The derivative of the product of f and g is

$$(\sec x \tan x)\cot x + \sec x(-\text{cosec}^2 x)$$
$$= \sec x - \sec x \,\text{cosec}^2 x \quad (\text{using } \cot x = 1/\tan x)$$
$$= \sec x(1 - \text{cosec}^2 x)$$
$$= \frac{1}{\cos x}\left[1 - \frac{1}{(\sin x)^2}\right] = \frac{(\sin^2 x - 1)}{\cos x \sin^2 x}$$
$$= \frac{-\cos^2 x}{\cos x \sin^2 x} = -\text{cosec}\, x \cot x$$

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On the other hand you may have realized that

$$\sec x \cot x = \frac{1}{\cos x}\frac{\cos x}{\sin x} = \operatorname{cosec} x$$

and from Table 1 the derivative of cosec x is $-\operatorname{cosec} x \cot x$. A little thought can often save a lot of hard work!

(f) Putting $f(x) = x^3$ and $g(x) = F(x)$ we have

$$\frac{d}{dx}\left[x^3F(x)\right] = 3x^2F(x) + x^3F'(x)$$

T13 There are several ways to tackle this question, but the following is probably the easiest.

$$v_x(t) = \frac{d}{dt}\left[s_x(t)\right] = \frac{d}{dt}\{[\cos(\omega t) - 2\sin(\omega t)]\exp(\alpha t)\}$$

From the product rule we can write this as

$$\begin{aligned}
&\frac{d}{dt}\{[\cos(\omega t) - 2\sin(\omega t)]\exp(\alpha t)\} \\
&\quad = \frac{d}{dt}\{[\cos(\omega t) - 2\sin(\omega t)]\} \times \exp(\alpha t) \\
&\qquad + [\cos(\omega t) - 2\sin(\omega t)]\frac{d}{dt}\exp(\alpha t) \\
&\quad = [-\omega\sin(\omega t) - 2\omega\cos(\omega t)]\exp(\alpha t) \\
&\qquad + \alpha[\cos(\omega t) - 2\sin(\omega t)]\exp(\alpha t)
\end{aligned}$$

so, $v_x(t) = \omega[-\sin(\omega t) - 2\cos(\omega t)]\exp(\alpha t) + \alpha s_x(t)$

Differentiating both sides of the last equation with respect to t we find

$$\begin{aligned}
a_x(t) &= \frac{d}{dt}v_x(t) \\
&= \frac{d}{dt}\{\omega[-\sin(\omega t) - 2\cos(\omega t)]\exp(\alpha t)\} + \alpha\frac{d}{dt}s_x(t) \\
&= \omega^2[-\cos(\omega t) + 2\sin(\omega t)]\exp(\alpha t) \\
&\quad + \omega\alpha[-\sin(\omega t) - 2\cos(\omega t)]\exp(\alpha t) + \alpha v_x(t)
\end{aligned}$$

We can now use the original specification for $s_x(t)$ to eliminate $[-\cos(\omega t) + 2\sin(\omega t)]\exp(\alpha t)$ from the above equation, and we can use the formula for $v_x(t)$ to eliminate $\omega[-\sin(\omega t) - 2\cos(\omega t)]\exp(\alpha t)$ from the same equation by writing

$$\omega[-\sin(\omega t) - 2\cos(\omega t)]\exp(\alpha t) = v_x(t) - \alpha s_x(t)$$

Consequently, we obtain

$$\begin{aligned}
a_x(t) &= -\omega^2 s_x(t) + \alpha[v_x(t) - s_x(t)] + \alpha v_x(t) \\
&= 2\alpha v_x(t) - (\omega^2 + \alpha^2)s_x(t)
\end{aligned}$$

T14 To make use of the product rule we need to write $f(x)g(x)h(x)$ as the product of the two functions $f(x)g(x)$ and $h(x)$. Then

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)h(x)] &= \frac{d}{dx}[f(x)g(x)]h(x) + f(x)g(x)\frac{d}{dx}h(x) \\
&= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)
\end{aligned}$$

T15 Take $f(x) = e^x$, $g(x) = \log_e x$ and $h(x) = \sin x$

then $f'gh + fg'h + fgh' = e^x \log_e x \sin x + e^x(1/x)\sin x + e^x \log_e x \cos x$

T16 (a) Set $f(x) = x$ and $g(x) = e^x$, to give $y = f(x)/g(x)$.

Now, $f'(x) = 1$ and $g'(x) = e^x$, so that the quotient rule gives us

$$\frac{dy}{dx} = \frac{1 \times e^x - x \times e^x}{(e^x)^2} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$

(b) Set $f(x) = 1 - x^2$ and $g(x) = \cos x$, to give $y = f(x)/g(x)$.

Now, $f'(x) = -2x$ and $g'(x) = -\sin x$, so that the quotient rule gives us

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(-2x)(\cos x) - (1-x^2)(-\sin x)}{\cos^2 x} \\
&= \frac{-2x\cos x + (1-x^2)\sin x}{\cos^2 x}
\end{aligned}$$

(c) Set $f(x) = x^5 - x^4 + x^3 - x^2$ and $g(x) = (x+1)^2$. To apply the quotient rule we need to differentiate $(x+1)^2$ which is easiest done if we expand it to obtain $x^2 + 2x + 1$.

Thus $f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x$ and $g'(x) = 2x + 2$. So,

$$\begin{aligned}
\frac{dy}{dx} &= [(5x^4 - 4x^3 + 3x^2 - 2x)(x+1)^2 \\
&\qquad - (x^5 - x^4 + x^3 - x^2)(2x+2)]/(x+1)^4 \\
&= [(5x^4 - 4x^3 + 3x^2 - 2x)(x+1) \\
&\qquad - 2(x^5 - x^4 + x^3 - x^2)]/(x+1)^3 \\
&= \frac{x(3x^4 + 3x^3 - 3x^2 + 3x - 2)}{(x+1)^3}
\end{aligned}$$

(d) With $f(x) = x\log_e x$, and $g(x) = \tan x$, we can apply the quotient rule. As an intermediate stage we differentiate $f(x)$ using the product rule.

$$\begin{aligned}
f'(x) &= x\frac{1}{x} + 1 \times \log_e x = 1 + \log_e x \\
g'(x) &= \sec^2 x \\
\frac{dy}{dx} &= \frac{(1 + \log_e x)\tan x + x\log_e x \sec^2 x}{\tan^2 x}
\end{aligned}$$

T17 (a) $f(x) - g(x) = (x-1)^2 - xe^x = x^2 - 2x + 1 - xe^x$

$$\begin{aligned}
\frac{d}{dx}[f(x) - g(x)] &= 2x - 2 - (xe^x + 1 \times e^x) \\
&= 2x - 2 - (x+1)e^x
\end{aligned}$$

(b) $$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \frac{d}{dx}[(x-1)^2 xe^x] \\
&= (2x-2)xe^x + (x-1)^2 e^x + (x-1)^2 xe^x \\
&= (x-1)e^x(x^2 + 2x - 1)
\end{aligned}$$

(c) $$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{(x-1)^2}{xe^x} \\
f'(x) &= (2x-2) = 2(x-1) \\
g'(x) &= xe^x + e^x = (x+1)e^x \\
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{2(x-1)xe^x - (x-1)^2(x+1)e^x}{x^2e^{2x}} \\
&= \frac{(x-1)(1 + 2x - x^2)}{x^2e^x}
\end{aligned}$$

(d) This can certainly be done using the quotient rule, but we can also obtain the answer from the reciprocal rule (i.e. Equation 13) and the solution to part (c) of this question as follows. First we let $F(x) = \dfrac{f(x)}{g(x)}$ so that, from the reciprocal rule

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$$\frac{d}{dx}\left[\frac{g(x)}{f(x)}\right] = \frac{d}{dx}\left[\frac{1}{F(x)}\right] = -\frac{F'(x)}{[F(x)]^2}$$

$$= -\frac{(x-1)(1+2x-x^2)}{x^2e^x} \Big/ \left[\frac{f(x)}{g(x)}\right]^2 \text{ (from part (c))}$$

$$= -\frac{(x-1)(1+2x-x^2)}{x^2e^x} \times \left[\frac{xe^x}{(x-1)^2}\right]^2$$

$$= \frac{(x^2-2x-1)e^x}{(x-1)^3}$$

T18 (a) Differentiating both sides of the following equation

$f(x)h(x) = 1$

using the product rule, we obtain

$$f'(x)h(x) + f(x)h'(x) = 0$$

Using the fact that $h(x) = \dfrac{1}{f(x)}$,

this can be rearranged to give

$$h'(x) = -\frac{f'(x)h(x)}{f(x)} = -\frac{f'(x)}{[f(x)]^2}$$

and, since $h'(x) = \dfrac{d}{dx}\left[\dfrac{1}{f(x)}\right]$, we have obtained the required result.

(b) Differentiating $f(x) \times \dfrac{1}{g(x)}$ using the product rule, we have

$$\frac{d}{dx}\left[f(x)\times\frac{1}{g(x)}\right] = f'(x)\times\frac{1}{g(x)} + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$= \frac{f'(x)}{g(x)} + f(x)\times\left\{-\frac{g'(x)}{[g(x)]^2}\right\}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

T19 (a) $f(x) = \sin x$ and $g(x) = \cos x$

so, $f'(x) = \cos x$ and $g'(x) = -\sin x$

thus $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

and, since $\cos^2 x + \sin^2 x = 1$, this gives

$$\frac{d}{dx}(\tan x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(b) To differentiate $\cot x = \dfrac{\cos x}{\sin x}$

let $f(x) = \cos x$ and $g(x) = \sin x$

so, $f'(x) = -\sin x$ and $g'(x) = \cos x$

Then

$$\frac{d}{dx}(\cot x) = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\text{cosec}^2 x$$

T20 (a) $f(x) = e^{-x} = \dfrac{1}{e^x} = \dfrac{1}{g(x)}$ where $g(x) = e^x$.

Since $g'(x) = e^x$ we have $f'(x) = -\dfrac{e^x}{(e^x)^2} = -\dfrac{1}{e^x} = -e^{-x}$

(b) $f(x)$ is a constant multiple of a sum of two functions, so, using our rules and the result of part (a),

$$\frac{df}{dx} = \frac{1}{2}(e^x - e^{-x})$$

(The function $\dfrac{1}{2}(e^x - e^{-x})$ is known as the *hyperbolic sine* of x and denoted by sinh x.)

$$[f(x)]^2 - \left(\frac{df}{dx}\right)^2 = \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$$

$$= \frac{1}{4}\left[(e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})\right] = 1$$

thus establishing the useful result that:

$$\cosh^2 x - \sinh^2 x = 1$$

T21 $F(x)$ is proportional to the reciprocal of $(x-a)^2$ and

$$\frac{d}{dx}(x-a)^2 = \frac{d}{dx}(x^2 - 2ax + a^2) = 2x - 2a = 2(x-a)$$

Hence, using the reciprocal rule,

$$F'(x) = -GMm\frac{2(x-a)}{(x-a)^4} = -\frac{2GMm}{(x-a)^3}$$

T22 (a) $\sec x = \dfrac{1}{\cos x}$

so $$\frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\text{cosec } x = \dfrac{1}{\sin x}$

so $$\frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{(-\cos x)}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\text{cosec } x \cot x$$

T23 You should have been able to establish this result using the quotient rule, but the following method, known as *implicit differentiation*, requires a little less algebra. (This method is explored more fully elsewhere in *FLAP*, see the *Glossary* for details.)

First write $y(1 + Ce^x) = Ce^x$

Differentiate both sides, using the product rule on the left-hand side

$$y'(1 + Ce^x) + yCe^x = Ce^x$$

Now eliminate $(1 + Ce^x)$ by writing $(1 + Ce^x) = \dfrac{Ce^x}{y}$, and divide the resulting equation through by Ce^x to get the required result

$$\frac{y'}{y} + y = 1 \quad \text{so} \quad y' = y(1-y)$$

(Notice that by using y' in this way we have avoided the need to differentiate y explicitly, hence the name *implicit differentiation*.)

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T24 $\dfrac{d}{dx}\left[(e^x + e^{-x})(e^x - e^{-x})\right] = \dfrac{d}{dx}\left(e^{2x} - e^{-2x}\right)$

$= 2e^{2x} + 2e^{-2x}$

T25

(a) $(\log_e 3)3^x$

(b) $(\log_e 3)3^{x-1}$ (if we write $3^{x-1} = 3^x/3$)

(c) $(\log_e 5)5^x$

(d) $\dfrac{d}{dx}\left(2^x e^x\right) = \dfrac{d}{dx}\left[(2e)^x\right] = [\log_e(2e)](2e)^x$

$= (1 + \log_e 2)2^x e^x$

T26 (a) $\dfrac{d}{dx}(\log_2 x) = \dfrac{1}{x \log_e 2}$

(b) $\dfrac{d}{dx}\left(\log_{10} x^2\right) = \dfrac{d}{dx}\left(2\log_{10} x\right) = \dfrac{2}{x \log_e 10}$

(c) The answer is zero, which is seen most easily by putting $\dfrac{\log_2 x}{\log_{10} x} = \log_2 10$, so we are simply required to differentiate a constant.

(d) $\dfrac{d}{dx}\left(a^x \log_a x\right) = \left(a^x \log_e a\right) \times \log_a x + a^x \times \dfrac{1}{x \log_e a}$

$= a^x\left(\log_e x + \dfrac{1}{x \log_e a}\right)$

if we use $(\log_e a) \times (\log_a x) = \log_e x$.

T27 Since $y(x) = \dfrac{\log_e 2 - x}{\log_e 2 + x}$ we have, from the quotient rule,

$$\frac{d}{dx}[y(x)] = \frac{d}{dx}\left(\frac{\log_e 2 - x}{\log_e 2 + x}\right)$$

$$= \frac{-1 \times (\log_e 2 + x) - (\log_e 2 - x) \times 1}{(\log_e 2 + x)^2}$$

$$= \frac{-2\log_e 2}{(\log_e 2 + x)^2}$$

It follows that

$$2(\log_e 2)y'(x) + [1 + y(x)]^2$$

$$= 2(\log_e 2)\frac{-2\log_e 2}{(\log_e 2 + x)^2} + \left(1 + \frac{\log_e 2 - x}{\log_e 2 + x}\right)^2$$

which (eventually) simplifies to zero, so the expression $y(x)$ satisfies the equation.

Exit test answers

E1 $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

For x positive and close to 0 the tangent has a large positive gradient. As x becomes increasingly positive, the gradient of the tangent decreases, approaching 0 as x approaches infinity. The graph of $y = \log_e x$ is given in Figure 8.

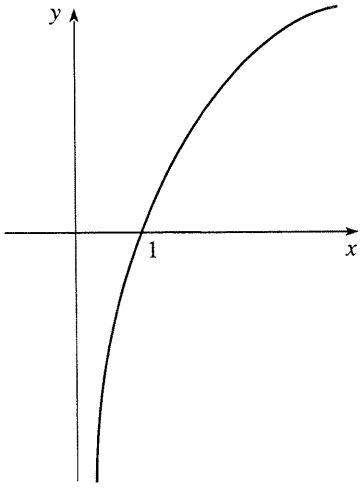


Figure 8 See Answer E1.

$\log_e x$ approaches $+\infty$ as $x \to \infty$ even though its rate of growth (its derivative) approaches 0.

(Reread Subsections 2.2 and 3.1 if you had difficulty with this question.)

E2 We need to evaluate the limit of

$$\frac{f(x+h) - f(x)}{h} \quad \text{as} \quad h \to \infty$$

Now $\dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{h}\left[\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}\right]$

$$= \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{(x+h)^2 x^2}\right] = \frac{1}{h}\left[\frac{-2hx - h^2}{(x+h)^2 x^2}\right] = \frac{-2x - h}{(x+h)^2 x^2}$$

As h approaches 0 the numerator approaches $-2x$ while the denominator approaches $x^2 \times x^2 = x^4$. Hence

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

(Reread Subsections 2.2 and 2.3 if you had difficulty with this question.)

E3 (a) $y = 2x(x-1)(x+2) = 2x(x^2 + x - 2) = 2x^3 + 2x^2 - 4x$

so $\dfrac{dy}{dx} = 6x^2 + 4x - 4$

(b) Using $\sin(A + B) = \sin A \cos B + \cos A \sin B$, we can rewrite the equation as

$y = \sin x \cos(\pi/4) + \cos x \sin(\pi/4)$

$$\frac{dy}{dx} = \cos x \cos(\pi/4) - \sin x \sin(\pi/4) = \frac{1}{\sqrt{2}}(\cos x - \sin x)$$

which is equal to $\cos\left(x + \dfrac{\pi}{4}\right)$

(c) $y = \log_e x^2 = 2\log_e x$ and therefore $\dfrac{dy}{dx} = \dfrac{2}{x}$

(Reread Subsection 3.1, 3.2 and 3.3 if you had difficulty with this question.)

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E4 (a) $f'(x) = \cos x + x(-\sin x) + 2x\sin x + x^2\cos x$

$= (1+x^2)\cos x + x\sin x$

(b) $\dfrac{d}{dx}\left(x - \dfrac{1}{x}\right)^2 = \dfrac{d}{dx}(x^2 - 2 + x^{-2}) = 2x - 2x^{-3}$

$= 2x(1 - x^{-4})$

So $f'(x) = 2x(1-x^{-4})e^x + \left(x - \dfrac{1}{x}\right)^2 e^x$

(c) $f'(x) = \dfrac{(1+2x)(1-x^2) - (1+x+x^2)(-2x)}{(1-x^2)^2}$

$= \dfrac{1+4x+x^2}{(1-x^2)^2}$

(d) $\dfrac{d}{dx}(\sqrt{x}\sin x) = \dfrac{1}{2}x^{-1/2}\sin x + \sqrt{x}\cos x$

So

$f'(x) = \dfrac{(\frac{1}{2}x^{-1/2}\sin x + \sqrt{x}\cos x)\log_e x - (\sqrt{x}\sin x)(1/x)}{(\log_e x)^2}$

$= \dfrac{1}{2\sqrt{x}}\dfrac{(\sin x + 2x\cos x)\log_e x - 2\sin x}{(\log_e x)^2}$

(e) $f(x) = \dfrac{e^x e^{\log_e x}}{\sqrt{x}} = \dfrac{xe^x}{\sqrt{x}} = \sqrt{x}e^x$

So $f'(x) = \dfrac{1}{2}x^{-1/2}e^x + \sqrt{x}\,e^x$

(f) $f'(x) = (\log_e a)a^x \times x^a + a^x \times ax^{a-1}$

$= (\log_e a)a^x \times x^a + a^{x+1}x^{a-1}$

(Reread Sections 3 and 4 if you had difficulty with this question.)

E5 We are told that

$$\frac{x^{n+1}-1}{x-1} = 1 + x + x^2 + \ldots + x^n$$

Differentiating the right-hand side gives the required series

$$1 + 2x + 3x^2 + \ldots + nx^{n-1}$$

which must be the same as the derivative of the left-hand side

$$\frac{d}{dx}\left(\frac{x^{n+1}-1}{x-1}\right) = \frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2}$$

$$= \frac{(n+1)x^{n+1} - (n+1)x^n - x^{n+1} + 1}{(x-1)^2}$$

$$= \frac{n\,x^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

(This question required thought in addition to a knowledge of differentiation. If you spotted the need to find the derivative but had difficulty finding the derivative, reread Subsections 3.1, 3.2 and 3.4.)

E6 It is certainly possible to differentiate both sides, using the reciprocal rule, and then to show that the results are equal. However, this result is merely the product rule when applied to the functions $1/f$ and $1/g$, and no working is necessary. (A salutary lesson confirming that a little thought can sometimes save a lot of work.)

(Reread Subsections 3.4 and 3.5 if you had difficulty with this question.)

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