AC circuits and electrical oscillations

Module P5.4

1 Opening items 1
2 AC circuits 3
2.1 Describing alternating currents 3
2.2 AC power and rms current 5
2.3 AC in resistors, capacitors and inductors 6
2.4 Resistance, reactance and impedance 10
2.5 The series LCR circuit 11
2.6 The parallel LCR circuit 14
2.7 Combining series and parallel circuits 15
2.8 Filter circuits 16
3 Oscillations in electrical circuits 17
3.1 Transient currents in an RC circuit 17
3.2 Transient currents in an LR circuit 19
3.3 Oscillations in LC circuits 20
3.4 Damped oscillations in LCR circuits 22
3.5 Driven oscillations in LCR circuits 23
4 Closing items 25
5 Answers and comments 28

1 Opening items

1.1 Module introduction

Perhaps before reading this module you should do a small electrical audit. Spend a minute or two wandering around your home looking at the electrical appliances which you find there. Most of them will have a small plate or sticker attached which tells you something about their electrical specifications. My toaster, which is quite a simple model, just says 240 V 1000 W 50 Hz. Whilst the back of my microwave — which incidentally was made in France — bears the more complicated legend

240 V ~ 50 Hz
fréq.: 2450 MHz
MW 1500 W/6,5 A

A good deal of this information concerns the sort of power supply that the appliance requires. In both the cases quoted the appliance requires a supply of *alternating current* (a.c.), with a *root-mean-square voltage* of 240 volts and a *frequency* of 50 hertz. Alternating currents are the central theme of this module. It seems clear that we must know something about them if we are to understand even the most basic information written on electrical appliances. (From 1995 the UK electricity supply is 230 volts.)

Section 2 begins by showing a mathematical way of describing alternating currents, and investigating their behaviour in various electronic circuit components. This involves building mathematical models of physical situations, a process that some students find daunting. To avoid confusion everything has been made as explicit as possible, so you may find some of the steps spelled out in rather a lot of detail. Other topics covered in Section 2 include *impedance* (the a.c. equivalent of resistance), and the construction of simple *filter circuits* that can be used to block signals in certain frequency ranges. *Phasor* methods are introduced and used in this discussion to determine the impedances (and other properties) of a variety of simple circuits.

Section 3 uses the results and concepts introduced in Section 2 to study the behaviour of charge, current and voltage in a variety of simple circuits. The circuits studied can exhibit various kinds of short-lived *transient* behaviour and, in some cases, sustained *oscillations* in which charges and currents flow back and forth repeatedly between different circuit components. In the most complicated cases this can even result in resonance when an alternating electrical supply excites an unusually large response from an appropriately designed circuit. This discussion of transients and oscillations also introduces the idea of a *differential equation* and some of the general principles that surround the use of such equations in particular physical contexts. However, no attempt is made to teach the mathematical techniques that are required to solve such equations in general.

Although most people have some understanding of direct current, it is alternating current which — in many ways — has the greatest bearing upon our lives. That is one of the reasons why this module is important since it deals with an aspect of electricity which impinges directly upon our everyday lives.

Study comment Having read the introduction you may feel that you are already familiar with the material covered in this module and that you do not need to study it. If so, try the *Fast track test questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track* questions? The answers are given in Section 5. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet all of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of these you should follow the

COPYRIGHT © 1995 THE OPEN UNIVERSITY
SUP 31509 9

guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 An inductor L of value 0.50 H, a resistor R of value 10 Ω and a capacitance C of value 5.07 μF are connected in series (an LCR circuit) to an alternating supply of frequency 100 Hz. What is the total impedance due to the three components? If the rms value of the supply voltage is 120 V calculate the *peak* value of the resulting current. What is the phase relationship between the current and the voltage?

Question F2 Starting from an appropriate condition concerning the total voltage drop around a circuit, show that the current $I(t)$ in a series LCR circuit that contains an external voltage supply $V_0 \sin(\Omega t)$ must satisfy an equation of the form:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I(t)}{C} = V_0\Omega\cos(\Omega t)$$

Describe the eventual behaviour of the general solution to this equation, after any transients have decayed.

1.3 Ready to study?

Study comment In order to study this module you will need to be familiar with the following terms: *amplitude*, *capacitance*, *charge*, *current*, *frequency*, *Kirchhoff's laws*, *parallel circuit*, *period*, *power*, *Ohm's law*, *oscillation*, *radians*, *resistance*, *series circuit*, *simple harmonic motion*, *vector* (and *vector addition*) and *voltage*. Mathematically, you should be familiar with the *trigonometric functions sin* (θ) and *cos* (θ) (including their *graphs*) and the use of the *inverse trigonometric function arctan* (x) to solve equations of the form $\tan(\theta) = x$. You will also need to know *Pythagoras's theorem* and to be familiar with *trigonometric identities*, including the following results:

$$\sin^2(\theta) + \cos^2(\theta) = 1, \quad \sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right), \quad \cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right),$$

and $$\cos(\theta) = -\sin\left(\theta - \frac{\pi}{2}\right)$$

In this module we write sin (θ) and cos (θ) rather than the more conventional sin θ and cos θ, since we are often concerned with the phase of the function and so wish to emphasize this.

You do not need to be fully conversant with *differentiation* in order to study this module, but you should be familiar with the *calculus notation* dx/dt used to represent the rate of change of x with respect to t. If you are uncertain about any of these definitions then you can review them now by reference to the *Glossary*, which will also indicate where in *FLAP* they are developed. In addition, you will need to be familiar with SI units. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics before embarking on this module.

Question R1 A 50 Ω resistor is connected to a 120 V d.c. supply. Calculate the current flowing in this resistor and hence the energy dissipated as heat in 5 min.

Question R2 A 100 μF capacitor is connected to a 6 V d.c. supply until it is fully charged. Calculate the maximum charge on the capacitor. If this capacitor is then discharged through a 10 Ω resistor, calculate the maximum current which will flow.

Question R3 A resistor and a capacitor are connected in *series* to a d.c. supply. Draw a diagram representing this electrical circuit. The two components are now re-connected in *parallel* with the same d.c. supply. Draw a diagram representing this new circuit.

Question R4 The displacement of a sinusoidal oscillation is described by the expression $y = A \sin(2\pi f t)$, where $A = 10$ cm and $f = 2$ Hz. Calculate the displacement y when $t = 0.25$, 0.30 and 0.70 s, respectively. What is the period of this oscillation?

You will find the answers to these questions in Section 5.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

2 AC circuits

2.1 Describing alternating currents and voltages

mains voltage
alternating current
peak value
period
phase
phase angle
frequency
angular frequency

You should already be familiar with the properties of a simple *d.c. circuit* of the kind shown in Figure 1. The box symbol represents a resistor, and the battery symbol represents a source of voltage (i.e. potential difference). The current I in the circuit will be such that the voltage drop V across the resistance R is equal to the voltage supplied by the battery. The voltage drop *across* the resistor can be measured using a voltmeter connected in *parallel* whilst the current flowing *through* the resistor can be measured using an ammeter connected in *series*. The resistance R can then be determined using *Ohm's law*

$$R = \frac{V}{I} \tag{1}$$

In analysing a d.c. circuit such as that of Figure 1 we generally think of the battery as a source of constant voltage, so that the current in any part of the circuit is a steady flow of electric charge in a single direction; that, after all, is the meaning of d.c. — *direct current*. However, not all voltage sources are constant and neither are all currents direct. For example, the **mains voltage** supplied by standard wall sockets in the UK varies rapidly with time, the potential difference between the 'live' and 'neutral' terminals changing not just in magnitude but even in sign 50 times a second. Not surprisingly, the currents that such varying voltages produce in circuits will also vary with time, generally altering both their magnitude and direction in response to the variations in the voltage. A current that periodically reverses its direction in this way is called an **alternating current** (a.c.).

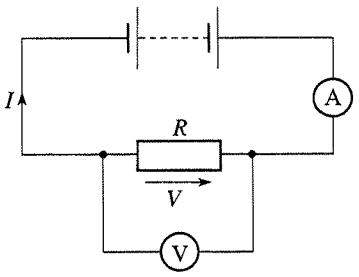


Figure 1 A simple d.c. circuit showing the measurement of potential difference and current using a voltmeter and ammeter, respectively. The arrow associated with the (conventional) current I indicates the direction in which *positive* charges would flow. The arrow associated with the voltage drop across the resistor points from low to high voltage and opposes the current direction. Negative values for I or V indicate currents and voltage drops that oppose the respective arrows.

Throughout this module we will neglect the *internal resistances* of batteries and other voltage sources.

The simplest kind of alternating current is represented in Figure 2. The current changes smoothly and regularly with time in a manner that may be described by a *sinusoidal* function (i.e. a sine or a cosine function), just like a simple harmonic oscillator. Its **peak value** (or *amplitude*) is I_0, and the **period** over which it goes through one complete oscillatory cycle is T. In drawing Figure 2 we have chosen the time $t = 0$ to be a moment at which the current is zero, consequently, it is marginally easier to represent this particular current by means of a sine function rather than a cosine function so we can write

$$I(t) = I_0 \sin\left(\frac{2\pi t}{T}\right) \tag{2}$$

Note that we have chosen to represent the current by $I(t)$ in order to emphasize that the value of I is changing with time and will generally depend on the value of t at which we choose to measure it.

Equation 2 is very similar in form to the equation used to describe the displacement of a *simple harmonic oscillator* from its equilibrium position. The quantity $2\pi t/T$, is called the **phase** of the oscillator, and increases by 2π each time t increases by an amount T. Since an increase of phase by 2π corresponds to a complete oscillation, the phase has many of the characteristics of an angle measured in radians and some authors prefer to treat it as an angle, in which case it may be called the **phase angle**.

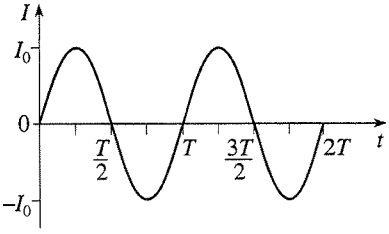


Figure 2 Two cycles of an alternating current with peak value I_0 and period T. Such a current may be represented by the function

$$I(t) = I_0 \sin\left(\frac{2\pi t}{T}\right).$$

It is possible to write the phase in Equation 2 in a couple of other ways. The **frequency** f of an oscillation is related to the period by $f = 1/T$, so it is always possible to replace $2\pi t/T$ by $2\pi f t$. More usefully we can introduce a related quantity called the **angular frequency** which is defined by $\omega = 2\pi/T$ so that the phase $2\pi t/T$ may be simply written as ωt. In terms of frequency and angular frequency, the alternating current described by Equation 2 may therefore be written in the following equivalent ways.

$$I(t) = I_0 \sin(2\pi f t) \tag{3a}$$

$$I(t) = I_0 \sin(\omega t) \tag{3b}$$

where $f = 1/T$ and $\omega = 2\pi/T$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Frequency and angular frequency are defined in such a way that either quantity may be measured in units of s^{-1}, thus the SI unit of each quantity is the hertz (Hz) since $1\ \text{Hz} = 1\ s^{-1}$.

So far, our mathematical descriptions of alternating currents have all been limited to the case where $I = 0$ when $t = 0$. Obviously this need not be the case; it is quite possible for an alternating current to have any one of its allowed values at $t = 0$. Figure 3 indicates the general situation that can arise. The current I_1 is of the kind we have already discussed and is described by:

$$I_1(t) = I_0 \sin(\omega t) \tag{4a}$$

but the current I_2, which has the same angular frequency and amplitude as I_1, has some other value at $t = 0$. This more general current may be represented by the expression

$$I_2(t) = I_0 \sin(\omega t + \phi) \tag{4b}$$

where the constant ϕ is called the **phase constant**. The phase constant is the value of the phase at $t = 0$ and determines the current at that time since $I_2(0) = I_0 \sin\phi$.

Given any two quantities that oscillate with the same angular frequency, such as I_1 and I_2, it is always possible to relate the oscillations of one to those of the other by describing the way in which the phase of one is related to that of the other. In the case of I_1 and I_2 we can describe this **phase relationship** by saying that I_2 **leads** I_1 by ϕ or that I_1 **lags** I_2 by ϕ. Note that ϕ could have many values, though it is customary to limit it to the range $-\pi < \phi \leq +\pi$ since adding or subtracting an integer multiple of 2π to the phase makes no difference to the value of the current.

phase constant
phase relationship
lead
lag
phasor

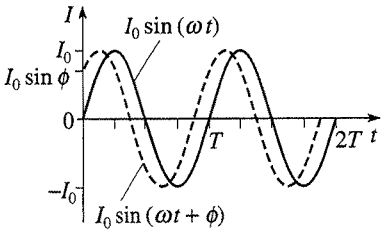


Figure 3 Sketch graph showing $I_1 = I_0 \sin(\omega t)$ and $I_2 = I_0 \sin(\omega t + \phi)$, in which I_2 leads I_1 by a phase constant ϕ.

◆ (a) Use the trigonometric identity $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$ to write down an expression that represents the current I shown in Figure 2 in terms of a cosine function.

(b) Use the trigonometric identity $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$ to determine the phase relationship between cos (ωt) and sin (ωt).

◇ (a) $I(t) = I_0 \sin\left(\frac{2\pi t}{T}\right) = I_0 \cos\left(\frac{2\pi t}{T} - \frac{\pi}{2}\right) = I_0 \cos\left(\omega t - \frac{\pi}{2}\right)$

(b) Using the given identity we see that cos (ωt) = sin $(\omega t + \pi/2)$, so it must be the case that the cos (ωt) *leads* sin (ωt) by $\pi/2$, or equivalently, that sin (ωt) *lags* cos (ωt) by $\pi/2$. This result will be very useful later in this module. □

The phasor representation

Another way of representing the alternating current of Figure 2 is in terms of a rotating **phasor**, as shown in Figure 4. A phasor is rather like the hand on a clock, though the phasors we will consider will all rotate in the *anticlockwise* direction. A phasor has a magnitude A and rotates at a fixed angular speed ω, so that the angle θ from the x-axis to the phasor increases with time. (☞) The y-component of the phasor (i.e. its 'projection' $y = A \sin\theta$ on to the y-axis) varies sinusoidally with θ, as

The magnitude A of a phasor is usually called its *amplitude*, and the angle θ its *phase*. The origin of these names will soon become clear!

The concept of a *phasor* is introduced more fully elsewhere in *FLAP*.

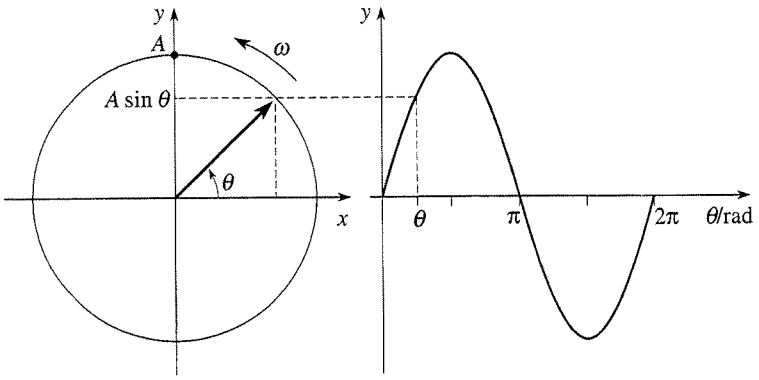


Figure 4 The projection of a phasor on to the y-axis is a sinusoidal function of the angle θ.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

indicated in Figure 4. It follows that a graph showing the variation of this y-component with time will be identical to Figure 2 provided the following requirements are met:

1 The magnitude A of the phasor is equal to the *amplitude* (or peak value) I_0 of the current.

2 At any time t, the angle θ of the phasor is equal to the *phase* (generally, $\omega t + \phi$) of the current.

instantaneous a.c. power
average a.c. power

Question T1 Draw a diagram showing the phasors that represent the alternating currents I_1 and I_2 (defined in Equations 4a and 4b) at time $t = 0$. In what way would your answer have changed if you had been asked to draw the phasors a quarter of a period later, at $t = T/4$? □

You will find the answers to all *Text questions* in Section 5.

The fact that oscillating systems, such as alternating currents, can be represented by phasors is interesting but little more than that at present. However, later in the module you will see that the phasor representation becomes very useful when we have to work out the combined effect of several alternating currents that differ in phase and amplitude.

2.2 AC power and rms current

When dealing with direct current, the *power* dissipated by a current passing through a resistor may be calculated using the formula

$$P = I^2R \tag{5}$$

Resistors can become hot as a result of this power dissipation when current passes through them.

When dealing with alternating current the situation is not so simple, since the current changes continuously. If the current at any *instant* is given by $I = I_0 \sin(\omega t)$, then the **instantaneous a.c. power** dissipation is given by

$$P = I^2R = I_0^2R\sin^2(\omega t)$$

This instantaneous power is shown in Figure 5. Though it is useful to know the expression for the instantaneous power it is not really of much direct relevance when dealing with everyday a.c. devices such as light bulbs and electric heaters. The frequency of the mains supply (50 Hz in the UK) is such that the instantaneous power rises and falls 100 times every second, which is much too fast for any heating effect to be seen or felt. When dealing with such everyday devices it is more useful to have an expression for the **average a.c. power** dissipated over a time that is much larger than the oscillation period of the current.

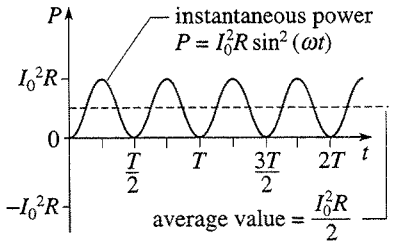


Figure 5 The instantaneous power dissipation $P = I_0^2R\sin^2(\omega t)$.

Over one complete period of oscillation T, the average value of an alternating current $I = I_0 \sin(\omega t)$ will be zero since every positive contribution to the average current will be counterbalanced by a negative contribution of equal magnitude. Nonetheless, an alternating current really does cause power dissipation in a resistor — you only have to switch on an electric fire to be sure of that. The reason for this is apparent from Equation 5; the instantaneous power is proportional to I^2, so the instantaneous power is positive whenever I is non-zero, irrespective of its sign. Thus, the average power $\langle P \rangle$ dissipated over one full period T will be equal to the mean of I^2 over that period. We may indicate this by using angular brackets to denote the mean and writing

The angular brackets $\langle\ \rangle$ represent the average value of the variable.

$$\langle P \rangle = \langle I_0^2R\sin^2(\omega t)\rangle = I_0^2R\langle\sin^2(\omega t)\rangle \tag{☞}$$

Note that we can take R, and I_0 if we wish, outside the angular brackets because they are constants.

Now the mean square current on the right-hand side can be worked out straightforwardly using the techniques of integration developed elsewhere in *FLAP*, but there is a simpler method that uses the special properties of the function $\sin^2(\omega t)$. Since the angular brackets indicate an average over a full period of oscillation, the time at which we start timing is unimportant, so we could equally well write

$$\langle P \rangle = \left\langle I_0^2R\sin^2\left(\omega t + \frac{\pi}{2}\right)\right\rangle = I_0^2R\langle\cos^2(\omega t)\rangle \tag{☞}$$

Remember, $\sin(\omega t + \pi/2) = \cos(\omega t)$

If we add the right-hand sides of these last two equations we find:

$$2\langle P \rangle = I_0^2R\langle\sin^2(\omega t)\rangle + I_0^2R\langle\cos^2(\omega t)\rangle = I_0^2R\langle\sin^2(\omega t) + \cos^2(\omega t)\rangle$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

root-mean-square current
rms current
root-mean-square voltage

However $\sin^2(\omega t) + \cos^2(\omega t) = 1$, which is a constant, so its average value over a full period is the same as its instantaneous value, i.e. 1.

Thus $\langle \sin^2(\omega t) + \cos^2(\omega t) \rangle = 1$

so, $2\langle P \rangle = I_0^2 R$

and $$\langle P \rangle = \frac{1}{2} I_0^2 R \qquad (6)$$

This amounts to saying that $\langle \sin^2(\omega t) \rangle = 1/2$, so the mean of I^2 over a full period is $I_0^2/2$.

As you can see, when an alternating current of peak value I_0 passes through a resistance R, the power dissipation is the same as that which would occur if a direct current of steady value $I_0/\sqrt{2}$ were to flow through the resistor. Because of the way it has been derived — by taking the square root of the mean of I^2 over a full period — this 'effective average current' is called the **root-mean-square current** or **rms current**. The root-mean-square current is usually denoted I_{rms} so we can write:

$$\langle P \rangle = I_{rms}^2 R \qquad (7a)$$

where $$I_{rms} = I_0/\sqrt{2} = 0.7071 I_0 \qquad (8a)$$

If we had started this investigation of the power dissipated by a resistor connected to an a.c. supply by considering an alternating voltage $V = V_0 \sin(\omega t)$ across the resistor rather than an alternating current through the resistor, we could have used the d.c. expression $P = V^2/R$ to lead us to the a.c. results:

$$\langle P \rangle = \frac{V_{rms}^2}{R} \qquad (7b)$$

where $$V_{rms} = V_0/\sqrt{2} = 0.7071 V_0 \qquad (8b)$$

In this case V_{rms} is the **root-mean-square voltage**.

Finally, note that by multiplying together the right-hand sides of Equations 7a and 7b, and then taking the square root, it can be seen that the average power dissipated by the resistor is also given by:

$$\langle P \rangle = V_{rms} I_{rms} \qquad (7c)$$

It is important to note that this equation only applies to *resistors*. A more general expression relating the average a.c. power dissipated in a circuit to the rms current and voltage will be given in Subsection 2.4.

Question T2 A 10 Ω resistor is supplied using a 6 V d.c. supply. Calculate the following quantities.

(a) The current flowing through the resistor.

(b) The energy dissipated every second in the resistor.

(c) The rms current and rms voltage which would produce identical energy dissipation with the resistor connected to an alternating supply.

(d) The corresponding peak values of alternating voltage and current. □

2.3 Alternating current in resistors, capacitors and inductors

In the last subsection we examined the *effect* of an alternating current passing through a resistor. In this subsection we look at the *origin* of such a current. In particular we will answer the following question, 'If an alternating voltage is applied across a resistor, what current will it cause to flow through the resistor?' As you will see, in the case of a resistor the relationship between the voltage and the current is quite straightforward, but we will also examine two other electronic components, a *capacitor* and an *inductor*, in which the relationship is not so simple.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Alternating current in resistors

Figure 6a shows a simple a.c. circuit that supplies an alternating voltage $V(t) = V_0 \sin(\omega t)$ across a resistor. Since Ohm's law $V = IR$ can be applied at any instant, we can expect that the varying voltage will produce a varying current given by

$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin(\omega t)$$

so, if we define $$I_0 = \frac{V_0}{R} \quad (9)$$

then we can write $$I(t) = I_0 \sin(\omega t) \quad (10)$$

A graph of this current, and the voltage that caused it, is shown in Figure 6b, along with the corresponding current and voltage phasors at an arbitrary time t. The peak current I_0 has been chosen for graphical convenience in Figure 6b; in practice it would be determined by the values of V_0 and R. The main point to notice about Figure 6b is that the current and the voltage rise and fall together, neither one leads or lags the other so there is no **phase difference**, and we can say that they are **in phase**. This is also clear from the phasors which rotate together, one on top of the other.

phase difference
in phase

The SI unit of resistance is the ohm (Ω), where $1\ \Omega = 1\ \mathrm{V\,A^{-1}}$.

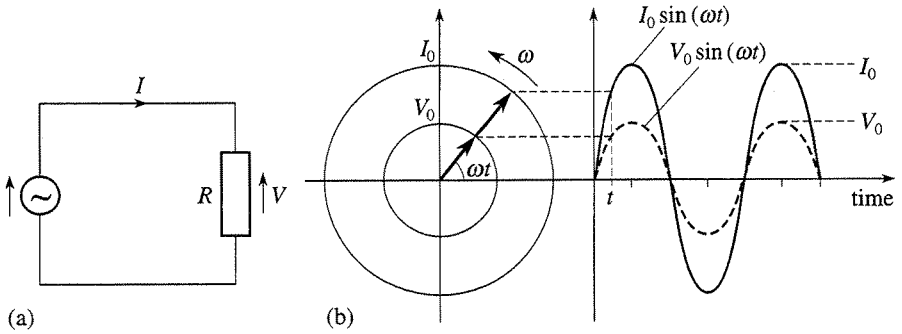


Figure 6 (a) A circuit supplying an alternating voltage (represented by the wavy line in a circle) across a resistor. The arrows shows the positive direction for the voltages, with an arrowhead at the positive end. (b) The current produced in the circuit is in phase with the voltage that causes it, as shown by the graph and the corresponding phasor diagram.

Alternating current in capacitors

A *capacitor* acts to store charge, and may generally be thought of as a pair of parallel conducting plates holding charges of equal magnitude but opposite sign. An alternating voltage $V(t) = V_0 \sin(\omega t)$ may be applied across a capacitor of fixed capacitance C by means of a circuit like that shown in Figure 7a. No conduction current actually passes *through* the capacitor in such a circuit but there is an ebb and flow of charge from the capacitor as it is alternately charged and discharged; it is this flow that constitutes the current in the circuit. Predicting this current is a slightly indirect process but the general approach is the same as before; we start with a result that can be expected to apply at any particular instant. In the case of a capacitor, the formula that we need is that which relates the charge q stored on the capacitor to the potential difference V across it: $q = CV$. Assuming this is true at every instant we can write

$$q(t) = CV(t) = CV_0 \sin(\omega t) \quad (11)$$

Since a current is equivalent to a flow of charge, a current will only flow in such a circuit while the charge on the capacitor is changing with time — that is while the capacitor is charging or discharging. The SI unit of capacitance is the farad (F), where $1\ \mathrm{F} = 1\ \mathrm{A\,s\,V^{-1}}$. The farad is a very large unit by everyday standards; capacitors of just a picofarad (10^{-12} F) or so are common.

Note that our (arbitrary) choice of direction for the arrow indicating the positive sense of voltages across the capacitor in Figure 7a means that the stored charge $q(t)$ must be interpreted as the charge on the *upper* plate of the capacitor. Whenever $V(t)$ is *negative* (implying that the upper plate is at the *lower* potential), $q(t)$ will also be *negative*.

Now, the current at any point in a circuit is determined by the rate of flow of charge at that point in the circuit. In Figure 7a the positive direction assigned to the current is such that a positive current will cause $q(t)$ to increase. So, using the derivative notation of differential calculus, we find

$$I(t) = \frac{dq(t)}{dt} \quad (12)$$

and combining Equations 11 and 12 gives us

$$I(t) = \frac{d[CV_0 \sin(\omega t)]}{dt}$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

This derivative can be easily evaluated by using the techniques of *differentiation* (or laboriously evaluated by drawing a graph of sin (ωt) and evaluating its gradient at many values of t). Using either method, the result will be

inductor
inductance
henry

$$I(t) = \omega C V_0 \cos(\omega t)$$

so if we define $$I_0 = \omega C V_0 \quad (13)$$

then we can write $$I(t) = I_0 \cos(\omega t) \quad (14)$$

which can be rewritten in terms of a sine function as

$$I(t) = I_0 \sin\left(\omega t + \frac{\pi}{2}\right) \quad (15)$$

Having expressed the current in terms of a sine function it is easy to see that in a circuit containing just a capacitor the current and the voltage are *not* in phase. The current *leads* the voltage by $\pi/2$. Figure 7b shows this relationship in graphical form, along with the associated phasors. As you can see the current phasor and the voltage phasor rotate at the same rate, but the current phasor is 90° ahead of the voltage phasor.

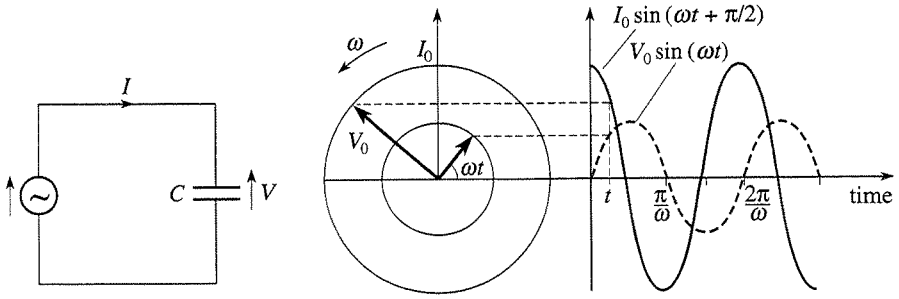


Figure 7 (a) A circuit supplying an alternating voltage across a capacitor. (b) The current produced in the circuit *leads* the voltage that causes it by $\pi/2$, as shown by the graph and the corresponding phasor diagram. As in Figure 6, the inner circle in part (b) represents the peak voltage V_0.

Alternating current in inductors

An **inductor** is essentially a coil of wire. When a current is made to flow through an inductor a magnetic field is produced in the coil. When the current is changed, the changing magnetic flux in the coil causes an induced voltage in the coil (*Faraday's law*) and the polarity of this voltage acts to oppose any change in the current. Figure 8a shows a simple circuit that applies an alternating voltage $V(t) = V_0 \sin(\omega t)$ across an ideal inductor. (☞) As the voltage changes the current will also tend to change. If the derivative dI/dt represents the rate of change of the current $I(t)$ passing through the inductor then the size of the voltage drop across the inductor at any instant will be

In practice inductors have resistance as well as inductance. However, for the time being, this will be ignored to keep things as simple as possible.

$$V(t) = L\frac{dI}{dt} \quad (16)$$

where the constant L is called the **inductance** of the inductor. You should be able to see from Equation 16 that the inductance can be expressed in units of V s A^{-1}; such a unit is called a **henry** (H), so 1 H = 1 V s A^{-1}. Widely used inductances vary from a few microhenry to hundreds of henry.

Aside Because inductors act to oppose changes in the current they are often said to be a source of back voltage which counters any change in the voltage supplied. Since the total voltage around a closed circuit must be zero (*Kirchhoff's law*) it follows that

supply voltage + back voltage = 0

If the supply voltage is given by $V(t)$, it follows from Equation 16 that the back voltage is

Many authors use the term *e.m.f.* (electromotive force) rather than voltage here. In *FLAP* we prefer to avoid the use of e.m.f. since the quantity is a voltage, not a force.

$$\text{induced voltage} = -L\frac{dI}{dt} \quad \square$$

Since $V(t) = V_0 \sin(\omega t)$ Equation 16 gives

$$\frac{dI}{dt} = \frac{V_0}{L}\sin(\omega t) \quad (17)$$

Equation 17 is a *first-order differential equation* of the kind that can be solved by the method of *separation of variables*. See the relevant headings in the *Glossary* for further details and references if you require them.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

differential equation

Equation 17 is an example of a **differential equation** since it involves the *derivative* dI/dt of the quantity that interests us, the varying current $I(t)$. The techniques needed to solve such equations are taught elsewhere in *FLAP* but you are not required to know them in order to study this module. Whenever such equations arise in this module we will simply quote their solutions and leave it to you to check by substitution that they are correct, should you wish to do so.

The solution to Equation 17, subject to the condition that there is no source of steady current in the circuit, is

$$I(t) = -\frac{V_0}{\omega L}\cos(\omega t)$$

so, if we define $I_0 = \dfrac{V_0}{\omega L}$ (18)

then we can write $I(t) = -I_0 \cos(\omega t)$

which we can rewrite in terms of a sine function as

$$I(t) = I_0 \sin\left(\omega t - \frac{\pi}{2}\right) \quad (19)$$

Having expressed the current in terms of a sine function we can again determine its phase relationship to the applied voltage $V(t)$. In this case the current *lags* the voltage by $\pi/2$. Figure 8b shows this relationship in graphical form, along with the associated phasors. The current phasor is 90° behind the voltage phasor.

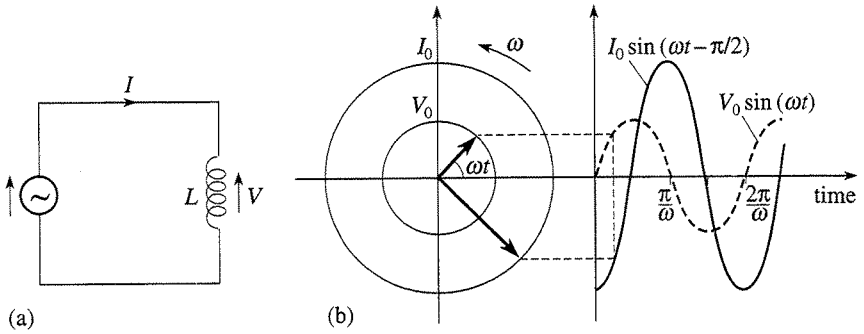


Figure 8 (a) A circuit supplying an alternating voltage across an inductor. (b) The current produced in the circuit *lags* the voltage that causes it by $\pi/2$, as shown by the graph and the corresponding phasor diagram.

Summing-up

As the above investigations show, when an alternating voltage $V(t)$ is applied across a resistor, a capacitor or an inductor the resulting current has the same frequency as the applied voltage but is not necessarily in phase with it. In the case of the resistor the phases do coincide, but in the capacitor the current leads the voltage by $\pi/2$, and in the inductor the current lags the voltage by $\pi/2$. (☞)

A useful way of remembering these phase relations is to use the mnemonic *CIVIL*. This tells us that in a *C*apacitor *I* leads *V* but *V* leads *I* in an inductor *L*.

The peak value of the current, I_0, depends on the resistance, capacitance or inductance in each respective case, but in addition, it also depends on the frequency of the supply in the case of a capacitor or an inductor. We will say more about this relationship in the next subsection.

◆ In each of the cases considered above we defined a quantity I_0 which was interpreted as the peak current: $I_0 = V_0/R$, $I_0 = V_0\omega C$ and $I_0 = V_0/(\omega L)$. Confirm that the quantity on the right-hand side of each definition can be expressed in units of ampere (A).

◇ In each case the unit of V_0 will be the volt (V). Since R is measured in ohm (Ω) and 1 Ω = 1 V A^{-1}, it follows that V_0/R may be measured in units of V/(V A^{-1}) = A. Similarly, since the unit of ω is s^{-1} and that of C is the farad (F) where 1 F = 1 A s V^{-1}, it follows that $V_0\omega C$ may be measured in units of V s^{-1} (A s V^{-1}) = A. Also, since L is measured in henry (H) and 1 H = 1 V s A^{-1}, it follows that $V_0/(\omega L)$ may be measured in units of V/(s^{-1} V s A^{-1}) = A. □

COPYRIGHT © 1995 THE OPEN UNIVERSITY

2.4 Resistance, reactance and impedance

reactive
reactance

The discussion of alternating currents in resistors in the last subsection shows no real surprises. In particular, if we measure the peak value V_0 of the voltage across a resistor and the peak value I_0 of the current that it causes to flow through the resistor (or the corresponding rms values), then we can easily determine the resistance R. It is given by

$$R = \frac{V_0}{I_0} = \frac{V_{\text{rms}}}{I_{\text{rms}}}$$

This resistance does not depend on the frequency of the a.c. supply and does not give rise to any phase difference between the voltage and the current.

In a similar way, we can use the quantity V_0/I_0 to define a sort of 'effective a.c. resistance' for capacitors and inductors. However, for these components the value of V_0/I_0 *does* depend on the frequency of the supply and there will be a phase difference between the voltage and the current. Since the response of capacitors and inductors depends on the nature of the supply they are said to be **reactive** components, and V_0/I_0 is called the **reactance** of a capacitor or an inductor, rather than its 'resistance'. Nonetheless, the unit of reactance, like that of resistance, is still the ohm (Ω). Using the symbols X_C and X_L to represent, respectively, the reactance of a capacitor and an inductor we see from Equations 13 and 18 (together with the relation $\omega = 2\pi f$) that

$$X_C = \left(\frac{V_0}{I_0}\right)_{\text{cap}} = \frac{1}{\omega C} \qquad (20)$$

and

$$X_L = \left(\frac{V_0}{I_0}\right)_{\text{ind}} = \omega L \qquad (21)$$

These results, along with the relevant phase relationships, are summarized in Table 1. Graphs showing how resistance and reactance vary with angular frequency are shown in Figure 9.

Table 1 Resistance/reactance of various circuit components, together with the phase relationship they produce between voltage and current.

Component	Resistance/reactance	Phase relationship
resistor	$R = V_0/I_0$ (constant)	V and I in phase
capacitor	$X_C = 1/(\omega C)$	I leads V by $\pi/2$
inductor	$X_L = \omega L$	I lags V by $\pi/2$

◆ Figure 9 indicates that the value of R is greater than the common reactance at which $X_L = X_R$. Must this always be the case, or does it depend on the particular resistors, capacitors and inductors we choose to consider?

◇ The three curves shown in Figure 9 have no direct connection with one another. They have been plotted on a single set of axes simply to save space. There is no reason why the values of R, C and L should be related in any way whatsoever, and therefore no reason to expect that the value of R will generally be greater than the common reactance. It is worth noting, however, that for any given values of C and L there *will* always be some particular frequency at which $X_C = X_L$. □

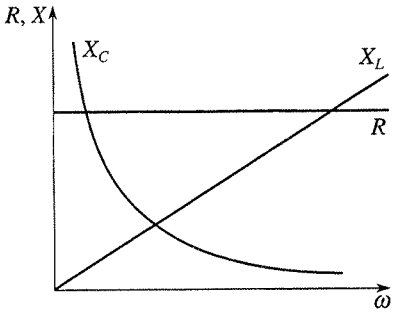


Figure 9 Graphs of R, X_C and X_L plotted as a function of angular frequency ω.

We can summarize this in words by saying that the frequency does not affect the conduction through a resistor but the conduction through a capacitor increases with frequency while that through an inductor decreases with frequency. For direct current the reactance of a capacitor becomes infinite while that for an inductor falls to zero. At very high frequencies the inductor reactance rises without limit while that for the capacitor falls to zero. Inductors tend to block high frequency currents while capacitors block d.c. currents.

Resistance and reactance are both special cases of a more general a.c. phenomenon known as *impedance*. If, as in Figure 10, an alternating voltage $V(t) = V_0 \sin(\omega t)$ is supplied to some general network of resistors, capacitors and inductors then the current $I(t)$ that the network draws from the supply will have the general form $I(t) = I_0 \sin(\omega t + \phi)$, where I_0 represents the peak value of the current and ϕ determines

COPYRIGHT © 1995 THE OPEN UNIVERSITY

the phase relationship between the voltage and the current. The values of I_0 and ϕ will depend on the resistances and reactances of the various components of the network, but in general terms we say that the **impedance** Z of the network is

impedance

$$Z = \frac{V_0}{I_0} = \frac{V_{rms}}{I_{rms}} \qquad (22)$$

If the network consisted of a single resistor, its impedance would simply be the resistance of that resistor and ϕ would be zero. Similarly, if the network consisted of a single capacitor or inductor then its impedance would equal the reactance of that capacitor or inductor and the value of ϕ would be $+\pi/2$ or $-\pi/2$, respectively. In the next two subsections we will investigate the impedances and phase relationships that arise from more complicated networks.

It is important to note that the instantaneous power supplied to the network in Figure 10 is given by

$$P(t) = V(t)I(t) = V_0 \sin(\omega t) I_0 \sin(\omega t + \phi)$$

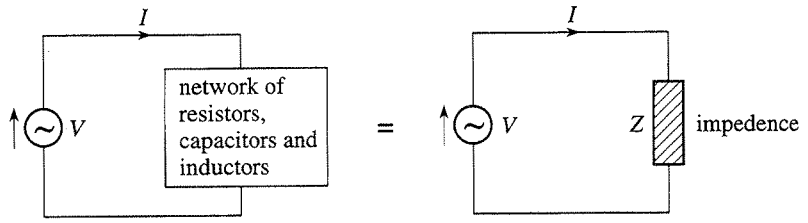


Figure 10 When a network of resistors, capacitors and inductors (with two external connections) is attached to a source of alternating voltage the effect is equivalent to that of a 'load' of impedance Z which causes the current to lead the voltage by a fixed phase difference ϕ.

In the case of a resistor (or even a network of resistors), where the phase difference $\phi = 0$, we have already seen that the average a.c. power over a full period ($T = 2\pi/\omega$) is $\langle P \rangle = V_0 I_0/2 = V_{rms} I_{rms}$. However, for a network containing capacitors and inductors ϕ will not generally be zero and it may be shown that

$$\langle P \rangle = V_{rms} I_{rms} \cos\phi \qquad (23)$$

Equation 23 may be derived by integration.

If you are familiar with *integration* you may know that the average can be written as

$$\langle P \rangle = \frac{1}{T}\int_0^T P(t)\,dt$$

where $P(t)$ is as defined above.

It follows that both the instantaneous and average power dissipated when the network consists of a single capacitor or a single inductor is zero since $\phi = \pi/2$ in these cases, and $\cos(\pi/2) = 0$.

◆ Express $\langle P \rangle$ for a network in terms of the impedance Z and the peak current I_0, and hence show that $\langle P \rangle = I_{rms}^2 Z \cos\phi$.

◇ $\langle P \rangle = V_{rms} I_{rms} \cos\phi = (V_0 I_0/2)\cos\phi = (I_0^2 Z/2)\cos\phi = I_{rms}^2 Z \cos\phi$ □

It is also true that for any network consisting only of pure inductors and capacitors, there is no power dissipation, since in each component the current and voltage are 90° out of phase. Power dissipation requires the pressure of resistance in the circuit.

2.5 The series LCR circuit

Combining C and R

Figure 11 shows a capacitor and a resistor connected in *series*. When circuit components are connected in this way the same current $I(t)$ flows through each of them, but the total instantaneous voltage $V(t)$ across them is the *sum* of the instantaneous voltages $V_R(t)$ and $V_C(t)$ across the individual components. (To this extent they behave like d.c. circuits.) Since the voltage across the resistor is in phase with the current while that across the capacitor is not, it follows that *the voltage across the capacitor cannot be in phase with the voltage across the resistor*. This means that the total voltage across the resistor and capacitor, which is just the applied voltage $V(t)$, will not be in phase with the voltage across *either* component. Nor therefore can the applied voltage be in phase with the current in the circuit. In what follows we are going to determine the relationship between the applied voltage and the current in this circuit, which will involve finding expressions for the ratio of their peaks $Z = V_0/I_0$ and for the phase difference ϕ between them.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

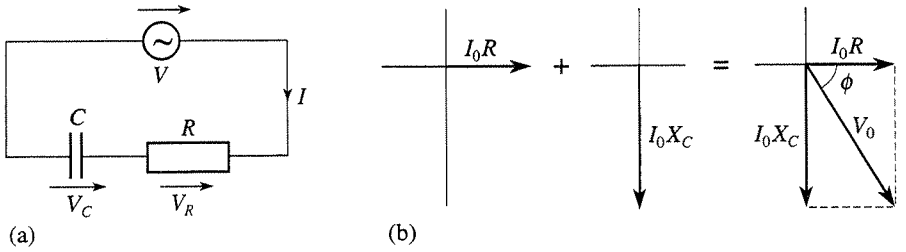


Figure 11 A capacitor and resistor in a series a.c. circuit (an RC circuit). The voltage phasors at $t = 0$ are shown, as is their sum which represents the total voltage that must be supplied at $t = 0$.

The easiest way to find the relationship between the peak voltages (and this is the key to finding the impedance) is to use phasor diagrams of the kind introduced in Subsection 2.1. Those needed in this case are also shown in Figure 11b. If the current is assumed to be of the form $I(t) = I_0 \sin(\omega t)$, then at $t = 0$ it can be represented by a horizontal phasor. (☞) The voltage across the resistor $V_R(t)$ is in phase with this current so it too can be represented by a horizontal phasor at $t = 0$. This voltage phasor is shown in Figure 11b, it has amplitude I_0R. Now, remembering that the voltage across a capacitor *lags* the current through it by $\pi/2$ (since the current *leads* the voltage by $\pi/2$) we see that at $t = 0$, the voltage across the capacitor is represented by a phasor that points vertically downwards. This phasor is also shown in Figure 11b, and has amplitude I_0X_C, where X_C is the reactance of the capacitor. The phasor representing the total voltage across the resistor and capacitor at $t = 0$ is obtained by adding the other two phasors together as though they were *vectors*. Using *Pythagoras's theorem*, the amplitude V_0 of this resultant phasor, which represents the total voltage across the circuit, is given by

Remember, the *phasor* representing an oscillating quantity is chosen so that its projection onto the vertical axis at any time t is equal to the value of the oscillating quantity at that time.

$$V_0^2 = I_0^2R^2 + I_0^2X_C^2$$

i.e. $$V_0^2 = I_0^2(R^2 + X_C^2)$$

so $$V_0 = I_0\sqrt{R^2 + X_C^2}$$

It follows that the impedance $Z = V_0/I_0$ of the series combination of a resistor and a capacitor is given by

$$Z = \sqrt{R^2 + X_C^2} \tag{24}$$

It can also be seen from Figure 11 that the total voltage lags the voltage in the resistor by ϕ. If we take the current phase as our reference zero then the voltage phase lag ϕ is given by

$$\tan\phi = -\frac{X_C}{R}$$

so $$\phi = -\arctan\left(\frac{X_C}{R}\right) \tag{25}$$

The convention is to give angles measured clockwise from the current phase zero a negative sign and anticlockwise angles a positive sign.

Since the voltage in the resistor is in phase with the current in the circuit, and since we chose to represent the current by $I(t) = I_0 \sin(\omega t)$, it follows that we can represent the total voltage by $V(t) = V_0 \sin(\omega t + \phi)$. So, in a series RC circuit, the impedance is $\sqrt{R^2 + X_C^2}$, and the current *leads* the applied voltage by $\arctan\left(\frac{X_C}{R}\right)$, since ϕ is a negative quantity in this case.

Combining L and R

The same thing can be done for a series combination of an inductor and a resistor — which is actually how a real inductor would behave. The relevant circuit is shown in Figure 12. Once again it will be assumed that the current is of the form $I(t) = I_0 \sin(\omega t)$, and as before we will write the applied voltage as $V(t) = V_0 \sin(\omega t - \phi)$. The challenge is to find the ratio V_0/I_0 and the value of ϕ that are appropriate to this case. Of course, V_0/I_0 can always be written as Z, so the first problem is to find the impedance Z of the series combination. We will use the voltage phasors at $t = 0$ to help solve this problem. The current in each series component is the same, and the voltage across the resistor is in phase with that current, so at $t = 0$ the phasor for the voltage across the resistor is horizontal and points to the right. The voltage across an inductor *leads* the current through it, so the phasor for the voltage across the inductor points vertically upwards in this case. The phasor representing the total voltage is the (phasor) sum of these two, as indicated in Figure 12.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

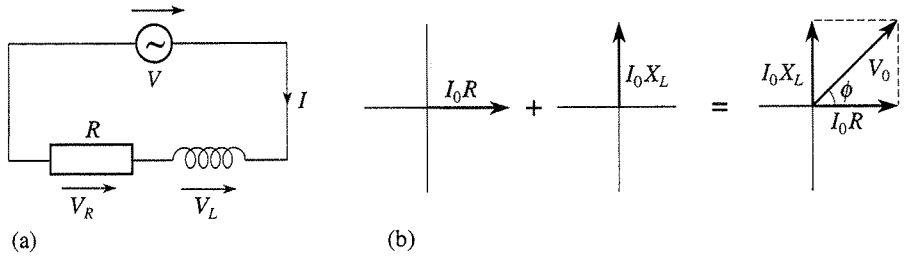


Figure 12 A resistor and an inductor in a series a.c. circuit (an LR circuit). The voltage phasors at $t = 0$ are shown, as is their sum which represents the total voltage that must be supplied at $t = 0$.

Following a similar argument to that used earlier, the impedance of the LR combination will be

$$Z = \sqrt{R^2 + X_L^2} \tag{26}$$

In this case the current (which is in phase with the resistor voltage) *lags* the total voltage. So $V(t) = V_0 \sin(\omega t + \phi)$ where in this case ϕ must be a *positive* quantity.

$$\phi = \arctan\left(\frac{X_L}{R}\right)$$

or equivalently $$\phi = \arctan\left(\frac{X_L}{R}\right) \tag{27}$$

So, in a series LR circuit, the impedance is $\sqrt{R^2 + X_L^2}$ and the current *lags* the applied voltage.

Combining L, C and R

Figure 13 shows a resistor, a capacitor and an inductor combined in a series a.c. circuit. The current in each component is the same and so has the same phase. It is sensible to take this current as defining our phase zero, with the voltages across each component having phase angles with respect to the current phase. This time we must add together three voltage phasors at $t = 0$. Fortunately two of them point in opposite directions along the same line, so their resultant is easy to determine, and this can be combined with the third phasor in the usual way.

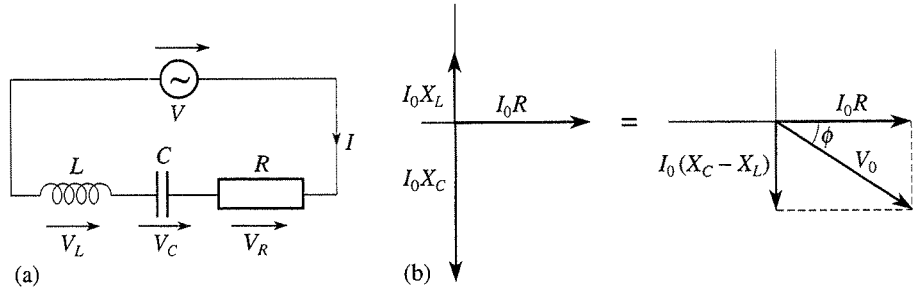


Figure 13 A series LCR a.c. circuit with the relevant phasor diagram showing how the individual voltages are combined. (In this case $X_C > X_L$).

Thus $$V_0^2 = I_0^2R^2 + I_0^2(X_L - X_C)^2$$

i.e. $$V_0^2 = I_0^2[R^2 + (X_L - X_C)^2]$$

but the impedance Z is given by $Z = V_0/I_0$, hence

the impedance of a *series LCR circuit* is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{28}$$

Substituting for the reactances ($X_L = \omega L$ and $X_C = 1/(\omega C)$) gives us

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{29}$$

It is also true that $Z^2 = R^2 + X^2$ where X is the total reactance of the circuit. So, for a series circuit

$$(\text{impedance})^2 = (\text{resistance})^2 + (\text{reactance})^2$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

The constant ϕ that determines the phase relationship between the current in the circuit and the supply voltage can also be deduced from the final phasor diagram in Figure 13. The value of ϕ is given by

$$\phi = \arctan\left(\frac{X_L - X_C}{R}\right) \qquad (30)$$

The results obtained earlier for RC and RL series circuits were special cases of the results contained in Equations 28 and 30.

These relationships would look rather different if we were to define the total voltage as the zero phase, with the current phase expressed in relation to this. This would be an equally valid representation and could be obtained from our representation simply by replacing ϕ by $-\phi$.

So, in a series LCR circuit, if X_C is greater than X_L then ϕ will be negative and the current in the circuit will lead the total voltage (as indicated in Figure 13). However, if X_C is less than X_L, Equation 30 will yield a positive value for ϕ, this corresponds to the case in which the current lags the total voltage. In this convention we are taking the conventional voltage in the resistor to define the zero phase condition, so $I(t) = I_0 \sin(\omega t)$ and the voltage applied to the whole circuit is then $V(t) = V_0 \sin(\omega t + \phi)$, where $V_0 = ZI_0$ and ϕ is given by Equation 30.

Question T3 A series LCR circuit has a supply current $I(t) = I_0 \sin(2\pi ft)$ where $I_0 = 0.1$ A and the supply frequency $f = 50$ Hz. If the components have values $R = 100\,\Omega$, $C = 50\,\mu$F (i.e. 50×10^{-6} F) and $L = 0.5$ H, calculate the impedance of the circuit and hence obtain an expression for the supply voltage. □

2.6 The parallel LCR circuit

The basic principle that underlies the analysis of a *parallel* LCR circuit, of the kind shown in Figure 14 is that the instantaneous voltage $V(t)$ is the same across each element but the current will generally differ from one component to another. When dealing with a *series* circuit, a phasor diagram was used to combine the *voltages* across each component, but with a *parallel* circuit, phasors must be used to combine the *currents* through each component. The total current being supplied to the combination of resistor, capacitor and inductor will therefore be the phasor sum of the individual currents in each component. The current in the resistor $I_R(t)$ is in phase with the supply voltage $V(t)$; the current in the capacitor $I_C(t)$ leads $V(t)$ by $\pi/2$, and the current in the inductor $I_L(t)$ lags $V(t)$ by $\pi/2$. Hence, if the supply voltage is given by $V(t) = V_0 \sin(\omega t)$ the currents in the resistor, capacitor and inductor at $t = 0$ can be represented by the phasors shown in Figure 14. These phasors may be added together (like vectors) giving a resultant phasor that represents the current $I(t)$ drawn from the supply.

Since it is now the voltage across the components which is the common feature, it makes sense to define the zero phase as being the phase of this voltage applied, allowing the current phases in each component to differ.

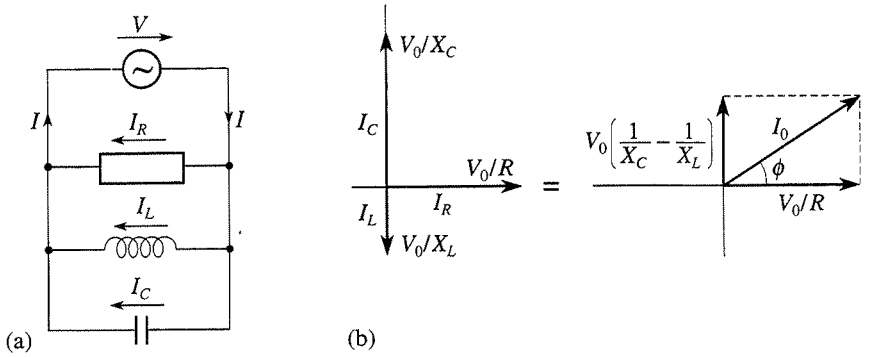


Figure 14 A parallel LCR circuit, the current phasors at $t = 0$, and the resultant phasor that represents the current drawn from the supply.

◆ In Figure 14 the amplitudes of the current phasors have been denoted by the constants V_0/X_C, V_0/R and V_0/X_L. This looks rather clumsy but makes more sense than denoting them I_C, I_R and I_L. Explain why.

◇ The symbols I_C, I_R and I_L have already been used to represent the currents in the capacitor, resistor and inductor; these currents vary with time and are represented at any moment by the (instantaneous) vertical component of the corresponding phasor. The quantities we are interested in are the constant amplitudes of the phasors. □

If we use Pythagoras's theorem to evaluate the amplitude of the resultant phasor in Figure 14, we find that the peak value of the total current in the circuit is given by

$$I_0 = \sqrt{V_0^2\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{V_0^2}{R^2}} \qquad (31)$$

so, $$\frac{I_0}{V_0} = \sqrt{\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}}$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

but the impedance $Z = V_0/I_0$. Hence;

the impedance of a *parallel LCR circuit* is given by

$$\frac{1}{Z} = \sqrt{\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}} \qquad (32)$$

Substituting for the reactances ($X_C = 1/(\omega C)$ and $X_L = \omega L$) we can rewrite this in the equivalent form

$$\frac{1}{Z} = \sqrt{\left(\omega C - \frac{1}{\omega L}\right)^2 + \frac{1}{R^2}} \qquad (33)$$

The constant ϕ that determines the phase relationship between the total current and the current through the resistor (and hence the voltage) may also be expressed in terms of the amplitudes of the three current phasors:

$$\phi = \arctan\left[\left(\frac{V_0}{X_C} - \frac{V_0}{X_L}\right) \Big/ \frac{V_0}{R}\right]$$

Dividing both the numerator (the top) and the denominator (the bottom) of this expression by V_0 gives us

$$\phi = \arctan\left[\left(\frac{1}{X_C} - \frac{1}{X_L}\right) \Big/ \frac{1}{R}\right] \qquad (34)$$

So, in a parallel LCR circuit, if $1/X_C$ is greater than $1/X_L$, then ϕ will be positive and the current will lead the voltage, but if $1/X_C$ is less than $1/X_L$, ϕ will be negative and the current will lag the voltage. In other words, if the voltage in the circuit is $V(t) = V_0 \sin(\omega t)$, the current will be given by $I(t) = I_0 \sin(\omega t + \phi)$, where $I_0 = V_0/Z$ and ϕ is given by Equation 34.

Question T4 A parallel LCR circuit has a supply voltage of the form $V = V_0 \sin(2\pi f t)$, where $V_0 = 100$ V and the supply frequency $f = 50$ Hz. If the components have values $R = 100\ \Omega$, $C = 50\ \mu$F and $L = 0.5$ H, calculate the impedance of the circuit and hence write an expression for the supply current. □

2.7 Combining series and parallel circuits

Using the rules that have been deduced above for calculating the impedance of components that are joined in series or in parallel (Equations 28 and 32, respectively) it is now possible for us to work out the impedance of more complicated networks. For example, in the case of the circuit shown in Figure 15 a capacitor and an inductor are joined in parallel, and that combination is connected in series with a resistor.

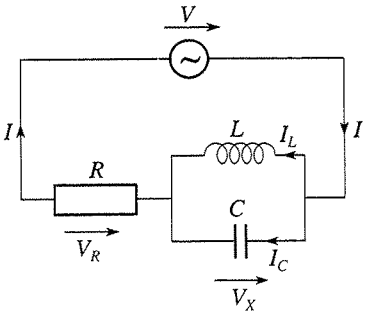


Figure 15 A circuit with a parallel combination of an inductor L and a capacitor C in series with a resistance R.

According to Equation 32, the parallel part of the circuit is equivalent to a single reactance X given by

$$\frac{1}{X} = \frac{1}{X_C} - \frac{1}{X_L}$$

which implies $$X = \frac{X_L X_C}{X_L - X_C}$$

and according to Equation 28 the total impedance Z of that reactance and the resistance with which it is in series is given by

$$Z^2 = R^2 + X^2$$

Thus, $$Z = \sqrt{R^2 + \left(\frac{X_L X_C}{X_L - X_C}\right)^2}$$

The voltage is the same across the inductor and the capacitor, whilst the (phasor) sum of the currents through the combination of inductor and capacitor is the same as the current through the resistor.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Question T5 A circuit like the one shown in Figure 15 has components with values R =10 Ω, L = 50 mH and C = 500 μF. If the supply voltage has V_{rms} = 12 V and f = 50 Hz, calculate the impedance of the circuit and hence the currents through, and voltages across, each part of the circuit. □

filter circuit
low-pass filter
high-pass filter

2.8 Filter circuits

It is important to remember that the impedance of reactive circuit components such as capacitors and inductors depends on the (angular) frequency of the supply. This property can be put to good use in various contexts, such as the construction of **filter circuits** designed to block signals in certain frequency ranges while allowing others to pass. Two simple circuits of this kind are shown in Figures 16 and 17.

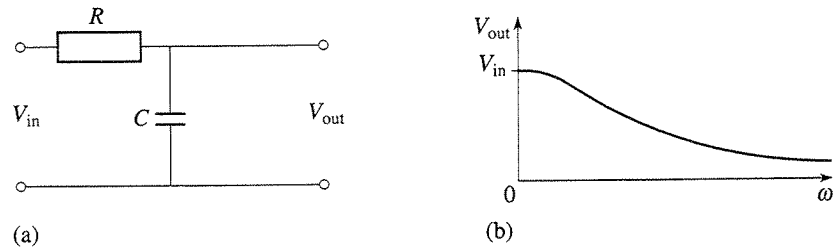


Figure 16 (a) A low-pass filter circuit. (b) Graph showing the rms output voltage as a function of the angular frequency of the input voltage.

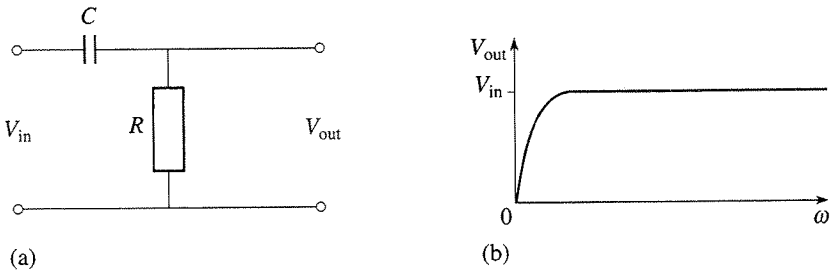


Figure 17 (a) A high-pass filter circuit. (b) Graph showing the rms output voltage as a function of the angular frequency of the input voltage.

In the case of Figure 16, an alternating supply of rms voltage V_{in} provides current to a capacitor and a resistor in series, and the varying voltage across the capacitor is used as an output voltage of rms value V_{out} that may supply some other circuit. The impedance of the series RC combination is $Z = \sqrt{R^2 + X_C^2}$ (Equation 12), so V_{in} is related to the rms current in the circuit by

$$V_{in} = I_{rms}\sqrt{R^2 + X_C^2}$$

but $V_{out} = I_{rms} X_C$

so $$V_{out} = V_{in}\frac{X_C}{\sqrt{R^2 + X_C^2}}$$

Now, this can be written in terms of angular frequency and capacitance as follows

$$V_{out} = V_{in}\frac{1/(\omega C)}{\sqrt{R^2 + [1/(\omega C)]^2}}$$

A graph showing the rms value V_{out} as a function of the angular frequency ω of the supply is shown in Figure 16b. As you can see, if the input voltage has a low frequency the output voltage will have a fairly similar rms value to V_{in}, but if the input has a high frequency the output will have a relatively small rms value. Thus, this particular circuit blocks high frequency signals and is therefore a crude example of a **low-pass filter**. You can easily imagine how useful such circuits might be in radios and other items of electrical equipment.

The next question invites you to find the formula that describes the graph in Figure 17 and hence to establish that the RC circuit shown there is an example of a **high-pass filter**.

Question T6 Find the rms voltage V_{out} for the circuit shown in Figure 17a, and confirm that its behaviour as a function of the input angular frequency is qualitatively similar to that shown in Figure 17b. □

COPYRIGHT © 1995 THE OPEN UNIVERSITY

3 Oscillations in electrical circuits

transient

You have now seen how capacitors and inductors respond to sinusoidal currents and voltages in a.c. circuits, but to have a more complete understanding of their behaviour you also need to know how they respond when supplies are switched on or off, or new connections are made. Such changes generally result in **transients** — that is brief bursts of current and voltage whose behaviour is characteristic of the component (or network of components) involved. We will temporarily return to d.c. circuits to consider this point.

3.1 Transient currents in an RC circuit

Figure 18a shows a capacitor being charged through a resistor by a d.c. supply. When the switch is first closed, at $t = 0$, the potential difference across the capacitor is initially zero, but it will increase with time as charge accumulates in the capacitor. This increasing potential difference $V_C(t)$ may be measured using a high impedance voltmeter (such as a *cathode ray oscilloscope*) and will change with time in the way indicated by Figure 18b. Eventually, the charge stored in the capacitor will build up to a maximum value $q_0 = V_0 C$, where V_0 is the voltage of the d.c. supply and C is the capacitance of the capacitor.

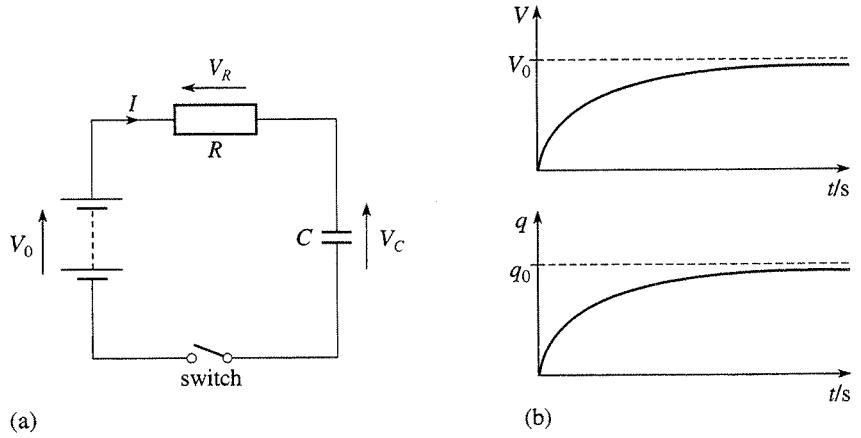


Figure 18 (a) An RC circuit for charging a capacitor through a resistor. (b) Graphs showing the change in potential difference, $V_C(t)$ and stored charge, $q(t)$ as a function of time.

If the capacitor in Figure 18a is fully charged, and the battery is then replaced by a piece of wire the capacitor will *discharge* through the resistor. The stored charge will then decrease with time in the way shown in Figure 19.

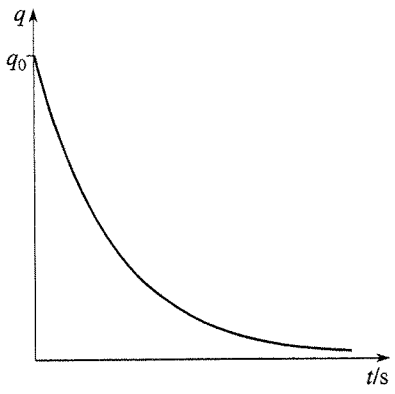


Figure 19 Discharge of a capacitor through a resistor allows the stored charge to fall from q_0 to 0.

Our aim now is to obtain the mathematical formulae that describe these curves.

Charging a capacitor

When charging the capacitor, the supplied voltage V_0 must equal the sum of the voltage drops across the resistor and the capacitor at any time t.

Note that the voltages across the resistor and the capacitor change with time, but their sum V_0 is constant.

So, $$V_0 = V_R(t) + V_C(t)$$

i.e. $$V_0 = I(t)R + \frac{q(t)}{C}$$

where $I(t)$ represents the current through the resistor at time t.

Thus $$I(t) = \frac{V_0}{R} - \frac{q(t)}{RC} \qquad (35)$$

Now, the current $I(t)$ is the rate of flow of charge that causes the stored charge $q(t)$ to increase, so we can write $I(t) = dq/dt$. Equation 35 can therefore be rewritten as

$$\frac{dq}{dt} = \frac{V_0}{R} - \frac{q(t)}{RC}$$

i.e. $$\frac{dq}{dt} = \frac{1}{RC}[V_0 C - q(t)]$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Using the fact that $q_0 = V_0 C$, and rearranging this last equation slightly gives us

$$\frac{dq}{dt} = -\frac{1}{RC}[q(t) - q_0] \tag{36}$$

first-order differential equation
boundary condition
initial condition

Now, R, C and q_0 are all constants, so what we have here is an equation which relates $q(t)$, the stored charge at any time t, to dq/dt, the instantaneous value of its rate of change. This is an example of a **first-order differential equation**, similar to the one we met in Subsection 2.3. It is said to be 'first-order' because it does not contain the second derivative d^2q/dt^2, nor any other derivative of q apart from the first derivative dq/dt. As before we will not work out the solution to the differential equation, we will just write it down and leave it to you to confirm that it is correct, should you wish to do so. In this case the general solution to the equation is

$$q(t) = q_0\left[1 - A\exp\left(-\frac{t}{RC}\right)\right] \tag{37}$$

Remember, $\exp\left(\frac{-t}{RC}\right) = e^{-t/RC}$ where e = 2.718 to three decimal places.

The arbitrary constant A that appears in this equation is a feature that arises in the general solution of every first-order differential equation. To determine its value in any particular case we must call on some appropriate item of additional information about the behaviour of $q(t)$; some property that is not implicit in Equation 36. This additional information is called a **boundary condition** or an **initial condition** of the solution. In the case of the charging capacitor a suitable condition is the requirement that at $t = 0$, when charging commences, there is no stored charge in the capacitor. In mathematical terms this means that $q(0) = 0$. Imposing this condition on Equation 37 (i.e. setting $t = 0$ and $q(0) = 0$) implies (☞)

Remembering that $e^0 = 1$.

$$A = 1$$

So, the particular solution to the differential equation that is relevant to us is:

$$q(t) = q_0\left[1 - \exp\left(-\frac{t}{RC}\right)\right] \tag{38}$$

Remember that the sign conventions adopted in Figure 18a mean that $q(t)$ will be the charge on the upper capacitor plate.

This is the mathematical description of the charging curve shown in Figure 18b.

Discharging a capacitor

When a charged capacitor is allowed to discharge through a resistor a changing current $I(t)$ flows through the circuit until the stored charge has been reduced to zero. In this case, since the stored charge is decreasing with time we know that its rate of change dq/dt will be negative. It follows that the current in the circuit $I(t) = dq/dt$ will also be a negative quantity, indicating that it flows anticlockwise, in opposition to the arrow shown in Figure 18a.

Now, in the absence of a battery $V_R(t) + V_C(t) = 0$

so $V_R(t) = -V_C(t)$

But, $V_R(t) = I(t)R = \frac{dq}{dt}R$ and $V_C(t) = \frac{q(t)}{C}$

Hence $$\frac{dq}{dt} = -\frac{q(t)}{RC} \tag{39}$$

Once again we have a first-order differential equation that may be solved by standard methods discussed elsewhere in *FLAP*. This time the general solution is

$$q(t) = A\exp\left(-\frac{t}{RC}\right)$$

where A is the arbitrary constant that we expect to arise in such a solution. The particular solution that is relevant to our problem is found by imposing the initial condition that the capacitor was charged at time $t = 0$, so $q(0) = q_0$. Imposing this condition we see that in this case $A = q_0$, so

$$q(t) = q_0\exp\left(-\frac{t}{RC}\right) \tag{40}$$

This equation shows the *exponential decay* expected from a discharging capacitor (Figure 19).

COPYRIGHT © 1995 THE OPEN UNIVERSITY

◆ Suppose that a capacitor is being discharged through a resistor as described above. If the charge stored on the capacitor at some particular time t_1 is q_1, what will be the remaining charge at time $t_1 + RC$?

◇ From Equation 40, $q(t_1) = q_0 \exp(-t_1/RC)$

and $q(t_1 + RC) = q_0 \exp[-(t_1 + RC)/RC]$

However, $q_0 \exp[-(t_1 + RC)/RC] = q_0 \exp(-t_1/RC) \exp(-RC/RC)$

$= q_0 \exp(-t_1/RC) \exp(-1)$

So, $q(t_1 + RC) = q(t_1) \exp(-1) = q_1 \exp(-1)$.

Thus, over any time interval of duration RC, the stored charge is reduced by a factor of 1/e. This is a characteristic property of exponential decay. □

Now that you know how a capacitor charges and discharges, the next thing to look at is the transient currents which flow when a circuit containing a resistor and an inductor is switched on and off.

3.2 Transient currents in an LR circuit

Figure 20a shows a resistor and an inductor connected in series to a d.c. supply of voltage V_0. At any time after the switch is closed, the total voltage across the circuit is given by the sum of the voltages across the individual components, so

$$V_0 = V_R(t) + V_L(t)$$

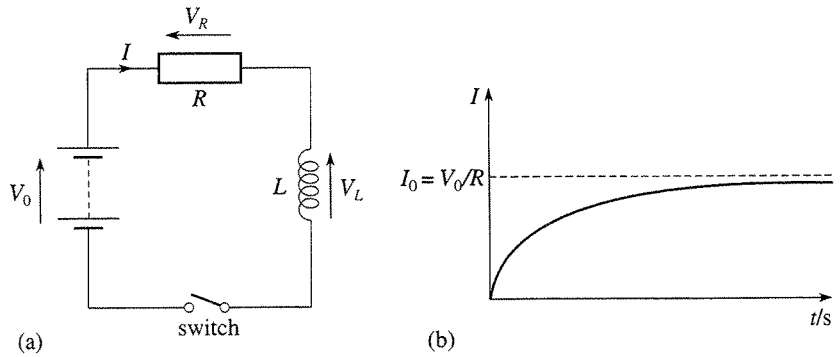


Figure 20 (a) An LR circuit for building the current through an inductor. (b) Graph showing the growth of the current as a function of time.

Using Ohm's law to find the voltage drop across the resistor and using Equation 16 for that across the inductor we can rewrite this in terms of the current $I(t)$

$$V_0 = I(t)R + L\frac{dI}{dt} \tag{41}$$

This can be rearranged to give

$$\frac{dI}{dt} = -\frac{R}{L}\left(I(t) - \frac{V_0}{R}\right) \tag{42}$$

Now, when this circuit has been switched on for a sufficiently long time the current will reach a final steady value $I(t) = I_0$. When that happens $dI/dt = 0$, so it follows from Equation 40 that $I_0 = V_0/R$. Thus, Equation 42 may be rewritten as

$$\frac{dI}{dt} = -\frac{R}{L}[I(t) - I_0] \tag{43}$$

◆ By comparing Equation 43 with Equation 36, write down its general solution.

◇ Equations 36 and 43 are of the same form except that q has been replaced by I, and $1/RC$ by R/L. It follows from Equation 37 that the general solution to Equation 43 must be

$$I(t) = I_0\left[1 - A\exp\left(-\frac{Rt}{L}\right)\right] \quad \square$$

◆ What is the value of A in this case, and what is the condition that determines it?

◇ If we require that the initial value of the current is zero (due to the opposition to change by the induced voltage of the inductor), we see that $I(0) = 0$. It follows that $A = 1$ in this case. □

COPYRIGHT © 1995 THE OPEN UNIVERSITY

The particular solution that interests us is therefore

$$I(t) = I_0\left[1 - \exp\left(-\frac{Rt}{L}\right)\right] \tag{44}$$

The transient current which flows in an inductor as the circuit is completed produces a curve similar to that in Figure 20b. Note that this curve for the growth of *current* in an inductor has the *same overall shape* as the curve for the growth of *stored charge* in a capacitor.

If the voltage supply is replaced by a piece of wire, then the current across the inductor will decay. In that case it follows from Equation 41 that

$$I(t)R = -L\frac{dI}{dt} \tag{45}$$

Question T7 By comparing Equation 45 with that used to describe the discharge of a capacitor, and by imposing an appropriate boundary condition, show that the decaying current can be described by Equation 46:

$$I(t) = I_0 \exp\left(-\frac{Rt}{L}\right) \tag{46}$$

Notice that in this case the time required for the current to decay by a factor of 1/e is L/R, so increasing the resistance will speed-up the decay.

3.3 Oscillations in LC circuits

Having looked at the response of capacitors and inductors individually to transient currents and voltages, the next step is to look at how they behave when connected together, as shown in Figure 21. In drawing the circuit, arbitrary 'positive' directions have been assigned to the current $I(t)$ and the potential differences $V_C(t)$ and $V_L(t)$. At all times the total voltage drop around the whole circuit must be zero, so

Remember that electrons are negatively charged and so flow in the *opposite* direction to 'conventional' current.

$$V_C(t) + V_L(t) = 0 \quad \text{so} \quad V_C(t) = -V_L(t)$$

where $V_C(t) = \dfrac{q(t)}{C}$ and $V_L(t) = L\dfrac{dI}{dt}$

Hence, $L\dfrac{dI}{dt} = -\dfrac{q(t)}{C}$ (47)

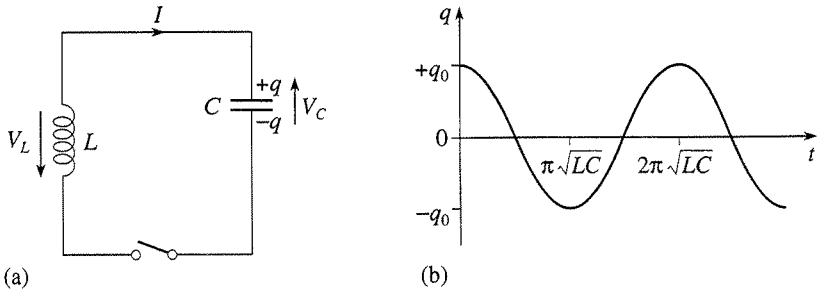


Figure 21 (a) A combination of a charged capacitor and an inductor (an LC circuit).
(b) The oscillating value of the stored charge in the capacitor.

If the capacitor in Figure 21a initially has a stored charge q_0 as shown, then, when the switch is first closed, current will flow around the circuit in the anticlockwise direction, so $I(t) = dq/dt$ will be negative and $q(t)$ will decrease. It follows from Equation 47 that dI/dt will be negative, causing $V_L(t)$ to be negative. This negative potential difference will counterbalance the positive (but decreasing) voltage $V_C(t)$ across the capacitor.

At all times $I(t) = \dfrac{dq}{dt}$

so $\dfrac{dI}{dt} = \dfrac{d^2q}{dt^2}$

and Equation 47 may consequently be rewritten as

$$\frac{d^2q}{dt^2} = -\frac{q(t)}{LC} \tag{48}$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Now, this is a differential equation that contains a second derivative, but no higher derivatives, so it is called a **second-order differential equation**. We can simplify Equation 48 a little, or at least make it more recognizable if we rewrite it in the following way:

$$\frac{d^2q}{dt^2} = -\omega_0^2 q(t) \tag{49}$$

where $$\omega_0 = \sqrt{\frac{1}{LC}} \tag{50}$$

second-order differential equation
natural frequency
natural angular frequency
simple harmonic oscillations

If you are familiar with simple harmonic motion, you may recognize Equation 49 as the sort of equation that describes an oscillating system such as a pendulum or a mass on a spring.

The general solution to a second-order differential equation contains *two* arbitrary constants. If we call them A and ϕ we can write the general solution as

$$q(t) = A\cos(\omega_0 t + \phi) \tag{51}$$

Equation 51 indicates that $q(t)$ oscillates with angular frequency ω_0, increasing and decreasing in repetitive cycles between $+A$ and $-A$, and having an initial value $A\cos(\phi)$ when $t = 0$. In order to determine the arbitrary constants A and ϕ we will need two appropriate boundary conditions. The first of these is that at $t = 0$ the stored charge in the capacitor is $q(0) = q_0$. The second is that this is the *maximum* positive value of the charge. (☞)

Other conditions describing the particular physical situation that interests us would serve equally well, e.g. the fact that the current is zero at $t = 0$.

The second of these conditions implies that $A\cos(\phi) = A$, so $\phi = 0$. The first condition then implies that $A\cos(\phi) = A\cos(0) = A = q_0$. Thus, the particular solution that is of interest to us is

$$q(t) = q_0\cos(\omega_0 t) = q_0\cos\left(\frac{t}{\sqrt{LC}}\right) \tag{52}$$

The oscillations of the stored charge described by this equation are shown in Figure 21b. Note that the period of the oscillations is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC} \tag{53}$$

so the corresponding frequency is

$$f_0 = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \tag{54}$$

This is usually called the **natural frequency** of the LC circuit, and ω_0 is called its **natural angular frequency**.

The charge oscillations described by Equation 52 cause the current in the circuit, $I(t) = dq/dt$, to oscillate and this causes associated oscillations in the voltages $V_C(t)$ and $V_L(t)$.

Question T8 Find the particular expressions for $I(t)$, $V_C(t)$ and $V_L(t)$ that correspond to Equation 52. Describe the phase relationship between each of these quantities and the stored charge $q(t)$. *Hint*: You may find the following mathematical results useful:

$$\frac{d}{dt}[\sin(\omega t)] = \omega\cos(\omega t), \quad \frac{d}{dt}[\cos(\omega t)] = -\omega\sin(\omega t);$$

$$\frac{d}{dt}[kf(t)] = k\frac{d}{dt}[f(t)] \quad \text{where } k \text{ is a constant.} \quad \square$$

The oscillations described in general by Equation 51, and in particular by Equation 52 are examples of **simple harmonic oscillations**. The same sort of oscillations can be seen in the *simple harmonic motion* (SHM) of many mechanical systems, such as a mass on a spring or a pendulum swinging back and forth through a small angle. The capacitor in an LC circuit, with its ability to store charge, behaves to some extent like a spring, a ready source of energy. The inductor, constantly striving to prevent changes in the current, is to some extent like a mass, always reluctant to be accelerated.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

As the charge in an LC circuit flows back and forth, the energy is ebbing and flowing between the capacitor and the inductor. At one moment in the cycle, all the energy is stored in the electric field of the capacitor. As this field decays, the energy is transferred to the inductor where it is stored in the magnetic field. As this field in turn decays, an electric field develops in the capacitor and the whole process continues its cycle. No power is dissipated in these interchanges because the voltages across the capacitor and the inductor always lead or lag the current by $\pi/2$. Thus in each case $\langle P \rangle = V_{rms} I_{rms} \cos \phi = 0$.

damping
damped oscillation

Question T9 Suppose that you have measured the currents and stored charges in an LC circuit in which the inductance is 1.00 H and the capacitance is 1.00 μF (i.e. 1.00×10^{-6} F), and that at time $t = 0$ you found that a current of 0.20 A flowed towards a capacitor plate that carried a charge of + 0.40 mC. Find the general expression for the current in the circuit at an arbitrary time t.

3.4 Damped oscillations in LCR circuits

The argument presented above showed that no energy is lost from an LC circuit, so the oscillations should continue for ever without any decrease in amplitude or change of frequency. However, in reality, electromagnetic energy *will* be lost due to heating because real inductors always have some resistance and this will alter the phase relationship between the current and the voltage. This phenomenon of decaying oscillatory behaviour is called **damping**. A more realistic circuit, containing a resistor as well as an inductor and a capacitor is shown in Figure 22. In this case the capacitor can be charged when the switch is set at A, and discharged through the inductor and the resistor when the switch is at B.

Damping is discussed generally in the *FLAP* modules which introduce *simple harmonic motion*.

Equivalent behaviour in a mechanical SHM process would be the transfer of energy between kinetic energy and gravitational potential energy in a pendulum.

◆ In what direction will the current flow when the switch is first moved from A to B? If $q(t)$ represents the charge on the upper plate of the capacitor, deduce the differential equation that $q(t)$ must satisfy.

The equivalent to resistive damping in a mechanical process could be air resistance, for example.

◇ The upper plate of the capacitor will be positively charged by the battery so when the capacitor starts to discharge through the resistor and inductor the (conventional) current will initially flow in the anticlockwise direction. If we arbitrarily choose the positive direction of current flow to be clockwise (so the initial current is *negative*), and if we take anticlockwise to be the positive direction for voltage drops, then we can write

$$V_C(t) + V_R(t) + V_L(t) = 0$$

where $V_C(t) = q(t)/C$; $V_R(t) = I(t)R = Rdq/dt$; $V_L(t) = LdI/dt = Ld^2q/dt^2$. Thus

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q(t)}{C} = 0 \qquad \square \ (55)$$

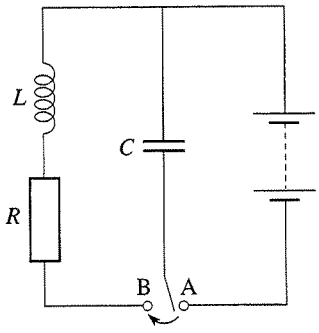


Figure 22 Circuit containing an inductor, a capacitor and a resistor. When the switch is moved from A to B the capacitor discharges through the resistor and the inductor.

The general solution to this second-order differential equation for values of R^2 which are less than $4L/C$ may be written:

$$q(t) = A\exp\left(-\frac{Rt}{2L}\right)\cos(\omega t + \phi) \qquad (56)$$

where

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (57)$$

As usual, boundary conditions (such as the values of $q(t)$ and $dq/dt = I(t)$ at some particular value of t) are required to determine the values of the arbitrary constants A and ϕ. However, irrespective of those values, provided A is not zero, it is possible to interpret Equation 56 as the mathematical description of a quantity that oscillates with angular frequency ω and with an 'amplitude' $A\exp\left(-\frac{Rt}{2L}\right)$ that decays exponentially with time. This kind of oscillation is an example of a **damped oscillation** and is illustrated by the dashed line in Figure 23. Note that if we identify $1/\sqrt{LC}$ as the natural angular frequency ω_0 in the case where $R = 0$, then we can write

$$\omega = \omega_0\sqrt{1 - \frac{R^2C}{4L}} \qquad (58)$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

showing that the angular frequency of a damped LCR circuit is lower than the natural frequency ω_0 of a pure (undamped) LC circuit.

The exponentially decaying damped oscillations that Equations 56 and 57 describe, those which arise for $R^2 < 4L/C$, are said to exhibit **underdamping**. Underdamping occurs with relatively small values of R. If R^2 is *very large* compared with $4L/C$ then the capacitor takes a relatively long time to discharge and no oscillation takes place. In this case, the circuit is said to exhibit **overdamping**. In **critical damping** $R^2 = 4L/C$ and the behaviour of $q(t)$ lies between these two extremes, with no full oscillation but, in general, with a single overshot of the $q = 0$ condition.

underdamping
overdamping
critical damping
driven oscillations
forced oscillations
resonance

Underdamping and *overdamping*, together with the related concepts of *light damping* and heavy damping are discussed in greater detail elsewhere in *FLAP*. See the *Glossary* for details.

3.5 Driven oscillations in LCR circuits

Whether they do so rapidly or slowly, oscillations in a resistive circuit (i.e. one which has a resistance) will always decay with time. However, if a supply of alternating current is made part of such a circuit it can supply the electromagnetic power that is dissipated and sustain the oscillations indefinitely. The oscillations that occur under these circumstances are called **driven oscillations** or **forced oscillations**, since their (angular) frequency will generally be that of the alternating supply rather than any 'natural' (angular) frequency associated with the circuit itself.

A driven LCR circuit is shown in Figure 24a. If the a.c. supply provides an alternating voltage $V(t) = V_0 \sin(\Omega t)$, then the sum of the voltages around the circuit must be zero and so the stored charge $q(t)$ must satisfy the second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q(t)}{C} = V_0 \sin(\Omega t) \qquad (59)$$

where Ω is the angular frequency of the supply.

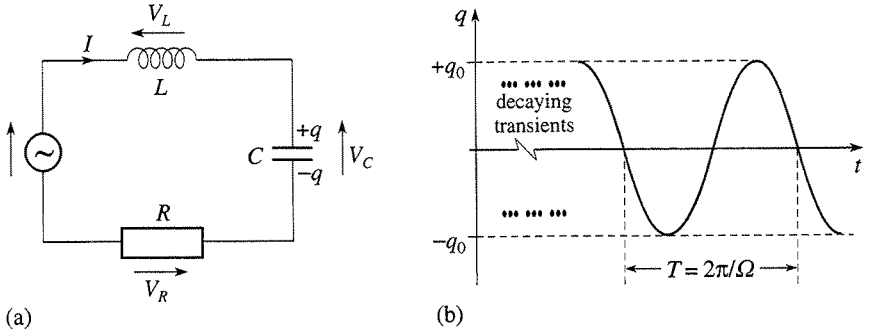


Figure 24 (a) A driven LCR circuit. (b) Graph showing the oscillatory behaviour of $q(t)$ after any transients have died away. The current in the circuit will show similar oscillations, though their amplitude will be $A\Omega$ and they will lead the charge oscillations by $\pi/2$.

This equation is harder to solve than those we dealt with earlier, so we will not bother to write down the general solution, we will simply round off our discussion of electrical oscillations by noting some of its main features.

The first thing to note is that after any initial transient behaviour has died away the solutions will take the form of simple harmonic oscillations described by

$$q(t) = A \sin(\Omega t + \phi)$$

where $$A = \frac{V_0}{\sqrt{[(1/C) - L\Omega^2]^2 + (R\Omega)^2}} \quad \text{and} \quad \phi = \arctan\left(\frac{-R\Omega}{(1/C) - L\Omega^2}\right)$$

Notice that the amplitude A of these driven oscillations depends on Ω, the angular frequency of the a.c. supply. A graph showing the variation of A with Ω is given in Figure 25b.

As you can see the amplitude in Figure 25b shows a pronounced maximum when Ω is close to the natural angular frequency $\omega_0 = 1/\sqrt{LC}$. This is the phenomenon of **resonance**, the existence of a driving frequency which provokes an exceptionally strong response from the circuit.

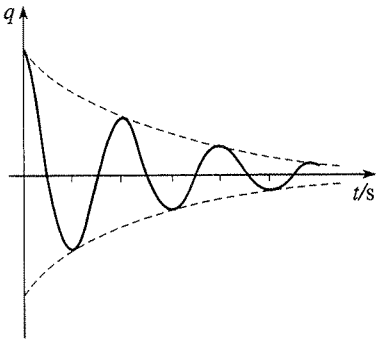


Figure 23 Damped oscillation in a series LCR circuit for which $R^2 < 4L/C$. Note that the value of $q(t)$ at $t = 0$ is related to the arbitrary constants, since $q(0) = A\cos\phi$

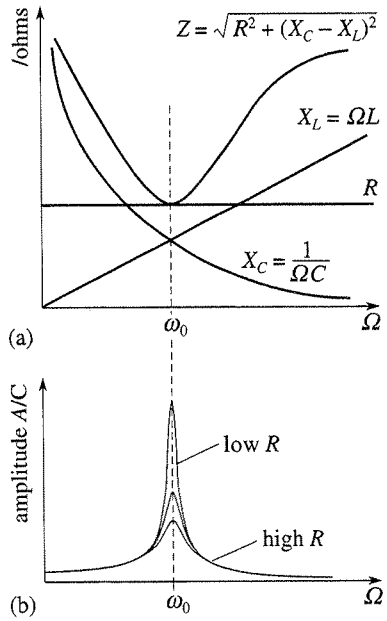


Figure 25 (a) Graphs showing the variation of reactance with angular frequency Ω. (b) A graph showing the variation of the amplitude of the oscillations with Ω.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

Since $\omega_0 = 1/\sqrt{LC}$ at resonance, or very nearly, we can rewrite the approximate resonance condition as

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{60}$$

that is, the reactances of the inductor and capacitor are equal

$$X_L = X_C \tag{61}$$

and the phase ϕ becomes zero.

This means that the impedance of a series LCR circuit

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

has its minimum value, R, at resonance (see Figure 25a). The rms current flowing in the circuit will be given by $I_{rms} = V_{rms}/Z$, so (for a constant rms supply voltage) the rms current will be a *maximum* at resonance in a series LCR circuit.

The voltage in a *parallel* LCR circuit (Figure 14) shows similar resonant behaviour. In that case

$$\frac{1}{Z} = \sqrt{\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}}$$

so, $1/Z$ is a *minimum* at resonance. Consequently Z has its maximum value R at resonance. The rms voltage across each component is the same in this case, and is related to the rms current in the circuit by $V_{rms} = Z I_{rms}$ so (for a constant rms supply current) the rms voltage is a maximum at resonance in a parallel LCR circuit.

4 Closing items

4.1 Module summary

1 An *alternating current* is a flow of electric charge that reverses its direction periodically. It may be described by

$$I(t) = I_0 \sin(\omega t + \phi) \tag{Eqn 3b}$$

where I_0 is the *amplitude* or *peak value* of the current, $(\omega t + \phi)$ is its *phase*, and ϕ is its *phase constant*. ω is the *angular frequency* of the current, which also has a *period* $T = 2\pi/\omega$ and a *frequency* $f = 1/T$. The *root-mean-square* value of such a current is given by $I_{rms} = I_0/\sqrt{2}$.

2 An alternating voltage $V(t) = V_0 \sin(\omega t)$ applied across a *resistor*, a *capacitor* or an *inductor* will cause an alternating current $I(t) = I_0 \sin(\omega t + \phi)$. The current through the resistor will be *in phase* with the voltage, so $\phi = 0$. The current through the capacitor will lead the voltage by $\phi = \pi/2$, and that through the inductor will *lag* the voltage by $\phi = \pi/2$.

3 The role of *impedance* in a.c. circuits is analogous to that of resistance in d.c. circuits. If a voltage $V(t) = V_0 \sin(\omega t)$ applied across a network of components causes an alternating current $I(t) = I_0 \sin(\omega t + \phi)$, the impedance of the network is given by $Z = V_0/I_0$, and the average power dissipated in the network is

$$\langle P \rangle = V_{rms} I_{rms} \cos\phi = I_{rms}^2 Z \cos\phi$$

4 A capacitor of capacitance C has a capacitative *reactance* $X_C = 1/(\omega C)$ where ω is the angular frequency of the voltage supply. Similarly, an inductor of inductance L has an inductive *reactance* $X_L = \omega L$. The impedance of a single capacitor or inductor is equal to its reactance, the impedance of a single resistor is equal to its resistance.

5 In a series LCR circuit the total impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{Eqn 28}$$

and the voltage leads the current by

$$\phi = \arctan\left(\frac{X_L - X_C}{R}\right) \tag{Eqn 30}$$

In a parallel LCR circuit the total impedance is

$$\frac{1}{Z} = \sqrt{\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}} \tag{Eqn 32}$$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

and the current leads the voltage by

$$\phi = \arctan\left[\left(\frac{1}{X_C} - \frac{1}{X_L}\right)\Big/\frac{1}{R}\right] \qquad \text{(Eqn 34)}$$

6 Short lived variations in current and voltage called *transients* can arise when the conditions in a reactive circuit are suddenly disturbed. Such transients may be studied with the aid of an appropriate *first-order differential equation*, relating the instantaneous value of a quantity (such as the stored charge $q(t)$) to its instantaneous rate of change (dq/dt). Such an equation has a general solution which contains an arbitrary constant. The general solution may be particularized to a specific physical situation by using an appropriate *boundary condition* to determine the value of the arbitrary constant.

7 In an LC circuit consisting of an capacitor connected to an inductor, the stored charge will exhibit *simple harmonic oscillations* described by

$$q(t) = A\cos(\omega_0 t + \phi)$$

where A and ϕ are arbitrary constants with values that must be determined from appropriate boundary conditions and the *natural angular frequency*

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad \text{(Eqn 50)}$$

Such oscillations satisfy a characteristic second-order differential equation which may be written in the form

$$\frac{d^2q}{dt^2} = -\omega_0^2 q(t) \qquad \text{(Eqn 49)}$$

8 The addition of a series resistor to an LC circuit results in an LCR circuit in which, under appropriate conditions, the stored charge exhibits *damped oscillations* in which the 'amplitude' decays with time. The frequency of these oscillations is less than the natural frequency of the corresponding undamped LC circuit.

9 The further addition of an alternating voltage supply to the LCR circuit eventually results in *driven oscillations* of the stored charge that have the same angular frequency as the supply. The amplitude of such oscillations is a function of the supply frequency and is strongly peaked when the supply frequency is close to the natural frequency of the corresponding LC circuit. The ability of a supply of appropriate frequency to excite large amplitude oscillations in the stored charge (and the current) is an example of the phenomenon of *resonance.*

4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Describe alternating currents and voltages in terms of sinusoidal functions.

A3 Characterize sinusoidally varying quantities in terms of their root-mean-square values and use those values in calculating power dissipation in simple a.c. circuits.

A4 Calculate the reactance of inductors and capacitors.

A5 Use phasor diagrams to determine current, voltage, impedance and phase in series and parallel LCR circuits, and in simple combinations of series and parallel LCR circuits.

A6 Describe the transient behaviour of stored charge (and consequently current) in RC and RL circuits, using appropriate first-order differential equations. (You are not expected to know how to find the general solutions to such equations.)

A7 Describe the simple, damped and driven oscillatory behaviour of stored charge (and consequently current) in LC, LCR and driven LCR circuits, using appropriate second-order differential equations. (You are not expected to know how to find the general solutions to such equations.)

A8 Describe the phenomenon of resonance and identify the conditions for resonance in circuits which display varying degrees of damping.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2*) An alternating current can be expressed as $I = I_0 \cos(\omega t - \pi/4)$. Rewrite this in terms of an equivalent sine function.

Question E2 (*A3* and *A4*) For a supply of rms voltage 100 V at 50 Hz, calculate the rms current which would flow in a series LC circuit when $C = 100\ \mu F$ and $L = 50$ mH.

Question E3 (*A3* and *A5*) A series LCR circuit with a rms supply voltage of 80 V and frequency 100 Hz contains a 50 Ω resistor, a 200 μF capacitor and an inductor of 75 mH. Draw a phasor diagram showing the voltage phasors across the three components and calculate the impedance of the circuit. Hence calculate the rms current flowing. How are the currents through the three components related to each other?

Question E4 (*A3* and *A6*) Describe the energy changes which take place when charge flows in a series a.c. circuit which contains *only* a capacitor and an inductor. If a resistor is now added to the circuit, what effect will this have on the energy changes?

Question E5 (*A7*) Starting from an appropriate condition concerning the total voltage drop around a circuit, show that the stored charge $q(t)$ in a series LCR circuit that contains no external supply must satisfy an equation of the form:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q(t)}{C} = 0$$

Describe the behaviour of the general solution to this equation in situations where the resistance R is small compared with $\sqrt{4L/C}$.

Question E6 (*A7*) For the circuit discussed in Question E5, sketch graphs showing the variation with time of the voltage across the capacitor for undamped and overdamped oscillations. State what conditions must be obeyed in each of these cases, and comment on the angular frequency of any oscillations that arise.

Question E7 (*A8*) Under what conditions will the current in an LCR circuit display resonance? Sketch a graph showing this effect and indicate the effect of increasing or decreasing the resistance in the circuit.

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

5 Answers and comments

Fast track answers

F1 The *impedance* of the series LCR circuit (see Figure 13) is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

where $\omega = 2\pi \times 100$ Hz, so

$$Z = \sqrt{(10)^2 + \left(2\pi \times 100 \times 0.5 - \frac{1 \times 10^6}{2\pi \times 100 \times 5.07}\right)^2}\ \Omega$$

$$= 10\,\Omega$$

So, $X_C \approx X_L$.

The rms voltage is related to the peak voltage by $V_{rms} = V_0/\sqrt{2}$. So in this case $V_0 = 169.7$ V.

But the peak current is given by $I_0 = V_0/Z$

so, $I_0 = 169.7$ V/10 Ω = 17.0 A

Because the total reactance $X_L - X_C$ is negligible in this case the current in the circuit will be in phase with the supply voltage.

F2 Following the sign conventions indicated in Figure 13, the appropriate condition on the voltage drops is

$$V_C(t) + V_R(t) + V_L(t) = V_0 \sin(\omega t)$$

But $V_C(t) = q(t)/C$; $V_R(t) = I(t)R$; $V_L(t) = LdI/dt$, where $q(t)$ represents the charge on the right-hand plate of the capacitor

So $$L\frac{dI}{dt} + RI(t) + \frac{q(t)}{C} = V_0 \sin(\Omega t)$$

Taking the derivative of both sides with respect to t, and using the rules

$$\frac{d}{dt}[k\,f(t)] = k\frac{df}{dt} \quad \text{and} \quad \frac{d}{dt}[\sin(kt)] = k\cos(kt)$$

where k is a constant and $f(t)$ a function of t, we find that

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = V_0\Omega\cos(\Omega t)$$

but, $dq/dt = I(t)$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I(t)}{C} = V_0\Omega\cos(\Omega t)$$

After transients have decayed the remaining oscillation will be excited by the driving voltage and will be of the form $I(t) = A\sin(\Omega t + \phi)$, where A and ϕ are constants whose values depend on Ω.

Ready to study answers

R1 Using *Ohm's law* $V = IR$, we find that the current flowing is

$I = 120$ V/50 Ω = 2.40 A

Power dissipation is given by

$P = VI = I^2R$

The total energy dissipated in time t is therefore

$E = Pt = I^2Rt$

i.e. $E = (2.40\text{ A})^2 \times 50\,\Omega \times (5 \times 60\text{ s}) = 86.4$ kJ

Consult *Ohm's law* and *power dissipation* in the *Glossary* for further information.

R2 The maximum *charge* on the *capacitor* is given by $q = VC$. Hence

$q = (6\text{ V} \times 100\ \mu\text{F}) = 600\ \mu\text{C}$

The maximum current flow is that which will flow initially. It is given by $I = V/R = (6\text{ V}/10\ \Omega) = 0.6$ A. The current, or rate of flow of charge, decreases thereafter.

Consult *charge* and *capacitor* in the *Glossary* for further information.

R3 The *series* and *parallel circuit diagrams* are shown in Figure 26a and b, respectively.

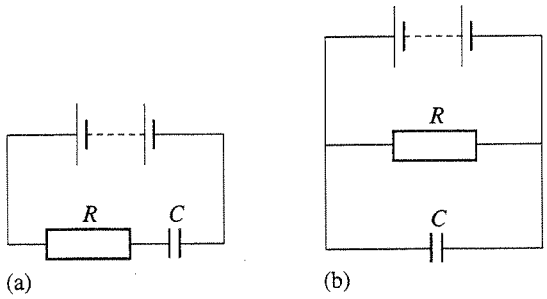


Figure 26 See Answer R3.

Consult *series* and *parallel circuit diagrams* in the *Glossary* for further information.

R4 Since $y = A\sin(2\pi f t)$, and given that the *amplitude* A is 10 cm and *frequency* f is 2 Hz, we can calculate the displacement at various times as follows. (Don't forget to use the radian mode on your calculator.)

When $t = 0.25$ s, $y = 10\text{ cm} \times \sin(2\pi \times 2\text{ s}^{-1} \times 0.25\text{ s})$

$= 10\text{ cm} \times \sin\pi = 0$ cm

Similarly, when $t = 0.30$ s, $y = -5.88$ cm, and when $t = 0.70$ s, $y = +5.88$ cm.

The *period* is related to the *frequency* by $T = 1/f$. Hence $T = 1/(2\text{ s}^{-1}) = 0.50$ s.

Consult the relevant terms in the *Glossary* for further information.

Text answers

T1 Phasors representing the alternating currents I_1 and I_2 at time $t = 0$ are shown in Figure 27. A quarter of a period later both phasors would have turned through 90° in the anticlockwise direction. Note that the phasors have been labelled by bold letters $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ to distinguish them from the currents which are represented by the vertical component of the corresponding phasor.

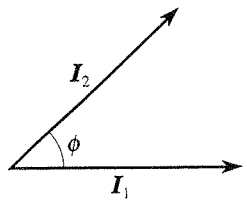


Figure 27 See Answer T1.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

T2 (a) $I = V/R = 6\text{ V}/10\ \Omega = 0.6\text{ A}$

(b) $P = VI = 6\text{ V} \times 0.6\text{ A} = 3.6\text{ W}$

(c) $I_{rms} = 0.6\text{ A}$ and $V_{rms} = 6\text{ V}$

(d) $V_0 = 6\text{ V}/0.7071 = 8.49\text{ V}$ and $I_0 = 0.6\text{ A}/0.7071 = 0.849\text{ A}$

T3 The impedance is given by Equation 28 or 29

$$Z = \sqrt{(100)^2 + \left(2\pi \times 50 \times 0.5 - \frac{1 \times 10^6}{2\pi \times 50 \times 50}\right)^2}\ \Omega$$

$$= 137\,\Omega$$

The phase constant ϕ that determines the amount by which $I(t)$ *leads* $V(t)$ is given by Equation 30

$$\phi = \arctan\left(\frac{2\pi \times 50\,\text{Hz} \times 0.5\,\text{H} - \dfrac{1}{2\pi \times 50\,\text{Hz} \times 50\,\mu\text{F}}}{100\,\Omega}\right)$$

$$= +0.75\,\text{radians}$$

(If you are using a calculator, use the radian mode.)

The positive value of ϕ indicates that $V(t)$ leads $I(t)$ in this case.

The peak of the supply voltage can be written $V_0 = ZI_0$, so the supply voltage can be expressed as

$$V(t) = (137\ \Omega)(0.1\text{ A})\sin(2\pi f t + 0.75)$$

i.e. $$V(t) = 13.7\sin(2\pi f t + 0.75)\text{ V}$$

T4 The impedance is given by Equation 32

$$\frac{1}{Z} = \sqrt{\frac{1}{(100)^2} + \left(\frac{2\pi \times 50 \times 50}{1 \times 10^6} - \frac{1}{2\pi \times 50 \times 0.5}\right)^2}\ \Omega^{-1}$$

Hence $Z = 73\Omega$.

The phase constant is given by Equation 34

$$\phi = \arctan\left(\frac{2\pi \times 50\,\text{Hz} \times 50\,\mu\text{F} - \dfrac{1}{2\pi \times 50\,\text{Hz} \times 0.5\,\text{H}}}{\dfrac{1}{100\,\Omega}}\right)$$

Hence $\phi = +0.75$ radians

So, the peak of the supply current can be written $I_0 = V_0/Z$ and the supply current can be expressed as

$$I(t) = \frac{100\sin(2\pi f t + 0.75)\text{ V}}{73\,\Omega}$$

Therefore, $I(t) = 1.4\sin(2\pi f t + 0.75)$ A.

The current leads the voltage by $\phi = 0.75$ radians

T5 The reactances of the inductor and capacitor, respectively, are

$$X_L = \omega L = (100\pi)\,\text{s}^{-1} \times (50 \times 10^{-3})\,\text{H} = 15.7\ \Omega$$

$$X_C = 1/(\omega C) = 1/[(100\pi)\,\text{s}^{-1} \times (500 \times 10^{-6})\,\text{F}] = 6.4\ \Omega$$

So the total reactance X of the parallel part of the circuit is given by $1/X = 1/X_C - 1/X_L$, from which $X = 10.7\ \Omega$.

The total impedance of the circuit is given by $Z^2 = R^2 + X^2$, from which $Z = 14.7\ \Omega$.

Using $V_{rms} = I_{rms}Z$, gives us $I_{rms} = 12\text{ V}/14.7\ \Omega = 0.82\text{ A}$.

Now, the rms voltages across each part of the circuit are given by

$$V_{R,\,rms} = I_{rms}R \quad \text{and} \quad V_{X,\,rms} = I_{rms}X$$

So, $$V_{R,\,rms} = 8.2\text{ V} \quad \text{and} \quad V_{X,\,rms} = 8.9\text{ V}$$

The rms currents in the two arms of the parallel part of the circuit are given by

$$I_{C,\,rms} = V_{X,\,rms}/X_C \quad \text{and} \quad I_{L,\,rms} = V_{X,\,rms}/X_L$$

So, $$I_{C,\,rms} = 1.39\text{ A} \quad \text{and} \quad I_{L,\,rms} = 0.57\text{ A}$$

As a consistency check on these results it is worth noting that

$$V_{rms}^2 = V_{R,\,rms}^2 + V_{X,\,rms}^2$$

which is to be expected because $V(t) = V_R(t) + V_X(t)$ but $V_R(t)$ and $V_X(t)$ differ in phase by $\pi/2$. Similarly

$$I_{rms} = I_{C,\,rms} - I_{L,\,rms}$$

which is to be expected since $I(t) = I_C(t) + I_L(t)$ but $I_C(t)$ and $I_L(t)$ are out of phase by π.

T6 Following similar steps to those used in analysing the low-pass filter we find

$$V_{out} = V_{in}\frac{R}{\sqrt{R^2 + X_C^2}} = V_{in}\frac{R}{\sqrt{R^2 + (1/(\omega C))^2}}$$

From this formula we can see that as ω approaches zero, $1/(\omega C) >> R$, and V_{out} approaches zero and that as ω becomes very large, $1/(\omega C) << R$, and V_{out} approaches V_{in}. Moreover, when $1/(\omega C)^2 = 3R^2$ (i.e. when $\omega = 0.58/RC$) $V_{out} = V_{in}/2$. This is the kind of graph shown in Figure 17b.

T7 Equation 45 would be identical to Equation 39 if we replaced I by q, and R/L by $1/RC$. It follows that the general solution to Equation 45 is

$$I(t) = A\exp\left(-\frac{Rt}{L}\right)$$

Requiring that $I(t)$ should have had its steady state value $I_0 = V_0/R$ at $t = 0$ implies that $A = I_0$ in this case, so that the particular solution of interest is that given by Equation 46, as required.

T8 $$I(t) = \frac{dq}{dt} = -\omega_0 A\sin(\omega_0 t + \phi)$$

but we know that $A = q_0$ and $\phi = 0$ in this case

so, $$I(t) = -\omega_0 q_0 \sin(\omega_0 t), \text{ where } \omega_0 = 1/\sqrt{LC}$$

$$V_C(t) = q(t)/C = (A/C)\cos(\omega_0 t + \phi) = (q_0/C)\cos(\omega_0 t)$$

$$V_L(t) = LdI/dt = -L\omega_0^2 q_0 \cos(\omega_0 t)$$

To reveal the phase relationships we can re-express these results as follows

$$q(t) = q_0\cos(\omega_0 t)$$

$$I(t) = \omega_0 q_0\cos(\omega_0 t + \pi/2)$$

$$V_C(t) = (q_0/C)\cos(\omega_0 t)$$

$$V_L(t) = L\omega_0^2 q_0\cos(\omega_0 t + \pi)$$

So, I *leads* q by $\pi/2$; V_C is in phase with q, and V_L *leads* q by π.

T9 We know that in general $q(t) = A\cos(\omega_0 t + \phi)$ (Eqn 51)

So, $$I(t) = \frac{dq}{dt} = -\omega_0 A\sin(\omega_0 t + \phi),$$

where $\omega_0 = 1/\sqrt{LC} = 10^3\,\text{s}^{-1}$

COPYRIGHT © 1995 THE OPEN UNIVERSITY

but at $t = 0$, $q(0) = A\cos\phi = 0.4 \times 10^{-3}$ C

and $I(0) = -A\omega_0 \sin\phi = 0.2$ A

so $$A\sin\phi = -\frac{0.2\text{A}}{\omega_0} = -0.2 \times 10^{-3}\text{ C}$$

Since $\sin^2\phi + \cos^2\phi = 1$, it follows that

$$A = \sqrt{(A\sin\phi)^2 + (A\cos\phi)^2}$$

$$= \sqrt{20 \times 10^{-8}}\text{ C} = 4.5 \times 10^{-4}\text{ C}$$

Also, since $\tan\phi = \sin\phi/\cos\phi$

$$\phi = \arctan\left(\frac{A\sin\phi}{A\cos\phi}\right) = \arctan(-0.5)$$

Now, care is needed at this point since arctan (x) is defined in such a way that its values are in the range $-\pi/2$ to $\pi/2$, but ϕ may be in the range $-\pi$ to π. In order to determine the true value of ϕ we note that cos ϕ is positive and sin ϕ is negative, implying that ϕ is in the range $-\pi/2$ to 0. Thus we may simply accept the value of ϕ provided by a calculator without needing to adjust it in any way.

Thus, ϕ = arctan (-0.5) = -0.46. (If you are using a calculator, use the radian mode.)

Hence, $I(t) = -0.45 \sin[(10^3\text{ s}^{-1})t - 0.46]$ A

Exit test answers

E1 Since cos (θ) = sin $(\theta + \pi/2)$, the expression given can be written as

$$I = I_0 \cos\left(\omega t - \frac{\pi}{4}\right) = I_0 \sin\left(\omega t - \frac{\pi}{4} + \frac{\pi}{2}\right)$$

Hence $$I = I_0 \sin\left(\omega t + \frac{\pi}{4}\right)$$

(If you had difficulty with this question, reread Subsection 2.1.)

E2 $I_{rms} = V_{rms}/Z$ and $Z = X_C - X_L$ for a series circuit containing only reactive elements. Furthermore $X_C = 1/(\omega C)$ and $X_L = \omega L$.

Now $X_L = (100\pi \times 50 \times 10^{-3})\,\Omega = 15.7\,\Omega$

and $X_C = 1/(100\pi \times 100 \times 10^{-6})\,\Omega = 31.8\,\Omega$

So $Z = (31.8 - 15.7)\,\Omega = 16.1\,\Omega$

and $I_{rms} = 100$ V/16.1 $\Omega = 6.20$ A

(If you had difficulty with this question, reread Subsections 2.2, 2.3, 2.4, 2.5 and 2.6.)

E3 The phasor diagram required is shown in Figure 28. The whole diagram may be rotated through an arbitrary angle.

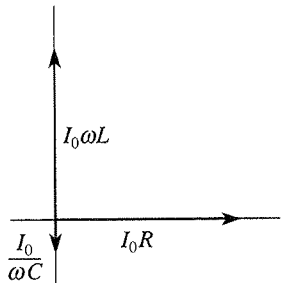


Figure 28 See Answer E3.

The impedance of a series LCR circuit is given by $Z = \sqrt{R^2 + (X_C - X_L)^2}$.

Using the same equations as in answer E2, X_L can be calculated as 47 Ω and X_C as 8 Ω.

Therefore $Z = \sqrt{50^2 + 39^2}\,\Omega = 63\,\Omega$

The rms current is given by

$I_{rms} = V_{rms}/Z = 80$ V/63 $\Omega = 1.26$ A

The current is the *same* through each component since they are connected in series.

(If you had difficulty with this question, reread subsections 2.4, 2.5 and 2.6.)

E4 In a series LC circuit, no energy is lost from the system. Energy ebbs and flows between the electric field in the capacitor and the magnetic field in the inductor, as charge oscillates around the circuit.

When the circuit contains a resistive element, electrical energy is dissipated as heat, and the electrical energy within the circuit decreases with time.

(If you had difficulty with this question, reread Subsections 3.1, 3.2, 3.3 and 3.4.)

E5 Following the sign conventions indicated in Figure 13, we find that the appropriate condition on the voltage drops is

$V_C(t) + V_R(t) + V_L(t) = 0$

But $V_C(t) = q(t)/C$; $V_R(t) = I(t)R = Rdq/dt$; $V_L(t) = LdI/dt = Ld^2q/dt^2$ where $q(t)$ represents the charge on the top capacitor plate towards which positive conventional currents flow.

So $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q(t)}{C} = 0$$

If R is small compared with $\sqrt{4L/C}$, the solutions to this equation describe (underdamped) oscillations with an amplitude that decays exponentially with time. The angular frequency of the oscillations is given by $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$.

(If you had difficulty with this question, reread Subsection 3.4.)

E6 The voltage across the capacitor is given by $V_C(t) = q(t)/C$, and we know from Question E5 that $q(t)$ exhibits damped oscillations. Graphs showing the variation of capacitor voltage against time for undamped (Figure 29a), and overdamped oscillations (Figure 29b) in a series LCR circuit are shown in Figure 29.

COPYRIGHT © 1995 THE OPEN UNIVERSITY

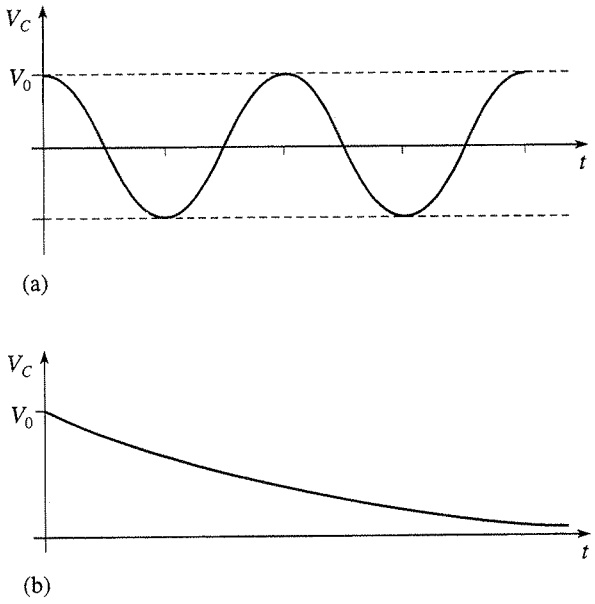


Figure 29 See Answer E6.

For undamped oscillations the circuit must have $R = 0$. The oscillation frequency is then given by $\omega_0 = \sqrt{1/(LC)}$.

For heavy overdamping, the resistance must be much larger than $\sqrt{4L/C}$ and there is no oscillation as such.

(If you had difficulty with this question, reread Subsections 3.3 and 3.4.)

E7 A series LCR circuit will display resonance if it is driven by an external voltage supply with an angular frequency Ω that is close to the natural angular frequency $\omega_0 = \sqrt{1/(LC)}$ of the circuit in the absence of driving or damping. Under these conditions

$$X_L = X_C \quad \text{and} \quad Z = \sqrt{R^2 + (X_C - X_L)^2} = R$$

At resonance the stored charge will be described by an expression of the form

$$q(t) = A \sin(\Omega t + \phi) \quad \text{so} \quad I(t) = \frac{dq}{dt} = A\Omega \cos(\Omega t + \phi)$$

where A and ϕ are constants whose values depend on Ω. The variation of A with Ω is shown in Figure 25b.

(If you had difficulty with this question, reread Subsection 3.5.)

COPYRIGHT © 1995 THE OPEN UNIVERSITY
S570modP5.4i1.1