Working with vectors

Module M2.5

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1 Opening items

1.1 Module introduction

When modelling the physical world mathematically we need to use quantities that can be measured. Such physical quantities mainly fall into two classes, *scalars* and *vectors*. The mathematics of scalars such as mass and temperature is simply the familiar mathematics of real numbers and functions, but the mathematics of vector quantities such as force and velocity, which have both a *magnitude* and a *direction*, is quite different, and is the subject of this module. Vectors are often introduced as geometric entities — more like arrows than numbers. Our main purpose in this module is to show you how the study of vectors may be transformed from geometry to algebra.

Section 2 of this module reviews the definition of scalar and vector quantities, the graphical (or geometric) representation of vectors, and the notation used in referring to vector quantities. (A more detailed introduction is given elsewhere in *FLAP*.) This section also reviews the basic operations of *scaling*, *vector addition* and *vector subtraction* in terms of the geometric representation. The section ends with a review of the *resolution* of vector quantities into *component vectors*, using the geometric representation.

Section 3 introduces the use of *unit vectors* to denote directions and, in particular, the use of $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ to specify the directions of the x-, y-, and z-axes in a right-handed Cartesian coordinate system. Section 3 continues by showing how these Cartesian unit vectors may be used to represent any vector in three dimensions, and then how vectors represented in *Cartesian form* may be manipulated in some of the basic operations of vector algebra, such as scaling and vector addition.

Section 4 introduces an alternative algebraic representation of vector quantities in terms of *ordered triples* of numbers. Sections 3 and 4 each conclude with a specific example which illustrates how, in practice, vector operations may be carried out using the Cartesian form, or the ordered triple representations, respectively.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 6. If you answer the questions successfully then you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ acting on a particle are such as to keep the particle in equilibrium. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are given as $\boldsymbol{F}_1 = (-\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k})$ newtons and $\boldsymbol{F}_2 = (-\boldsymbol{i} - 5\boldsymbol{k})$ newtons.

(a) Find the vector $\boldsymbol{F}_3$.

(b) Suppose the magnitude of $\boldsymbol{F}_1$ is doubled, and the magnitude of $\boldsymbol{F}_3$ is tripled (without changing their directions), while $\boldsymbol{F}_2$ remains unchanged. Find the new resultant force acting on the particle, the magnitude of this resultant force, and the unit vector acting in the direction of this resultant force.

Question F2 In the absence of an air current, the velocity of a particle is given by $\boldsymbol{v}_p = (2, 2, -1)$ m s^{-1}. Find the resultant velocity of the particle, and the time taken for the particle to travel a distance of 5 m along its resultant path, when an air current of velocity $\boldsymbol{v}_c = (-1, 1, 3)$ m s^{-1} is present.

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1.3 Ready to study?

Study comment Although the basic concepts of *scalar* and *vector* quantities are reviewed briefly in this module, it is assumed that you have some knowledge of these already. For example, you should be familiar with the definitions and methods of representation of *scalar* and *vector quantities*, and terms such as *zero vector*. You should understand the ideas of the *scaling of a vector quantity* by a scalar, the *addition* and *subtraction of vectors*, and the resolution of vectors into *component vectors*. Also, you should be familiar with the terms: *acceleration*, *displacement*, *electric charge*, *mass*, *momentum*, *temperature* and *velocity*. In addition, it is assumed that you have met *Newton's laws of motion*, and that you appreciate the conditions required for a particle to be in *equilibrium*. You should be familiar with basic mathematical concepts including the ideas of *Cartesian coordinate systems*, *Pythagoras's theorem* and the use of *trigonometric functions*. You should understand also the ways in which directions may be given with respect to points of the compass. If you are uncertain about any of these terms then you can review them by reference to the *Glossary* which will indicate also where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether you need to review some of the topics before embarking on this module.

vector
geometric vector
triangle rule
vector sum
resultant vector

Question R1 A right-angled triangle ABC has sides AB and BC of length 12 cm and 5 cm, respectively. Angle $A\hat{B}C$ is 90°. Determine the angles $B\hat{A}C$ and $A\hat{C}B$ and the length of the side AC.

Question R2 A ladder 5 m long stands on level ground with its highest point against a wall at a point 4.5 m above the ground. Determine the angle of inclination of the ladder to the ground, and the distance of the foot of the ladder from the bottom of the wall.

Question R3 Points A and B are given by their Cartesian coordinates in two dimensions as (1, 2) and (4, 3), respectively. Determine the lengths OA, OB and AB, and the angle $A\hat{O}B$, where O is the origin of the coordinate system.

You will find the answers to these questions in Section 6.

2 A brief review of scalars and vectors

2.1 What are scalars and vectors?

A **vector** is a mathematical object that has both *magnitude* and *direction*. Vectors may be represented diagramatically by *arrows* or *directed line segments*, as shown in Figure 1 where the magnitude and direction of a vector are indicated by the length and orientation of an arrow or line segment. These graphical representations of vectors are sometimes referred to as **geometric vectors**.

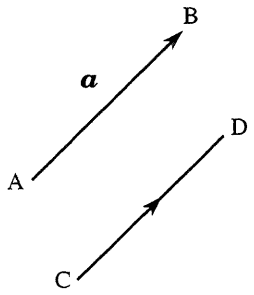


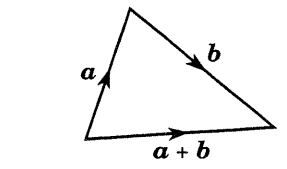
Figure 1 Vector $\boldsymbol{a}$ or $\overrightarrow{AB}$ is represented by an arrow, and $\overrightarrow{CD}$ by a directed line segment. Notice that while the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same magnitude, they do not have the same direction, and they are therefore distinct vectors.

It is customary to use distinctive notation when referring to vectors in order to emphasize the importance of their direction as well as their magnitude. For example, the vector from A to B in Figure 1 can be denoted by $\overrightarrow{AB}$ and, similarly, the vector from C to D can be denoted $\overrightarrow{CD}$. Alternatively, and more commonly, we might denote the vector $\overrightarrow{AB}$ by $\boldsymbol{a}$, where the bold typeface warns us that $\boldsymbol{a}$ is a vector with direction as well as magnitude.

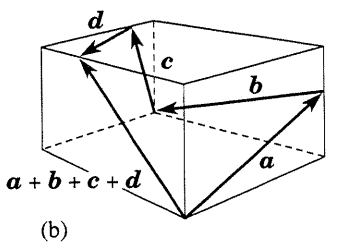
It should be noted that geometric vectors are completely specified by their magnitude and direction so $\overrightarrow{AB} = \overrightarrow{CD}$, even though they have been drawn at different points on the page.

The sum of two vectors is determined by the **triangle rule**, see Figure 2a, and the **vector sum** $\boldsymbol{a} + \boldsymbol{b}$ is often known as the **resultant vector**. The result of adding any number of vectors, such as $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ in Figure 2b, is obtained by simply joining them 'nose to tail'.

While 'geometric' vectors are valuable concepts, their practical use is limited by the difficulty of drawing such objects in three dimensions, and the major purpose of this module is to show you how a geometric construction like that shown in Figure 2b for the sum of four 'geometric' vectors can be replaced by a simple form of algebra.



(a)



(b)

Figure 2 (a) The sum of two vectors obtained from the triangle rule. (b) The sum of four vectors (in three dimensions).

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Scalar quantities

Many physical quantities can be specified completely by a single number together with an appropriate unit of measurement, for example length, mass, time and temperature. Such quantities are called **scalar quantities** or simply **scalars**, and we commonly denote them by symbols such as L, m, t and T. (☞)

scalar quantities
scalars
magnitude of a vector
vector quantity
scaling
unit vector

Vector quantities

There are many physical quantities, for example *displacement*, *velocity*, *acceleration* and *force*, which cannot be completely defined by their magnitudes (in the appropriate units); each of them is only really specified if we are also told the direction in which they act. The essence of such quantities is that they have both magnitude *and* direction, where the **magnitude** is a non-negative scalar quantity which gives the 'size' of the quantity.

The dimensions (or units) of the quantities on each side of an equation involving scalars must, of course, be the same. Thus two lengths can be added to give a total length, but the addition of mass to length would have no physical meaning.

> **Vector quantities** are physical quantities that have both magnitude *and* direction (and which can be adequately represented by geometric vectors).

Notation

In this module (and generally in *FLAP*) we will follow the standard convention of using bold type to indicate vectors, but in handwritten work it is usual to indicate a vector, $\boldsymbol{a}$, by a wavy underline, $\underset{\sim}{a}$, (☞) while a geometric vector or displacement vector from A to B may be written $\overrightarrow{AB}$.

$\underset{\sim}{a}$ is often used to indicate bold type to a printer. You may see alternative notation used elsewhere, e.g. $\underline{a}$, $\underline{AB}$, $\overline{AB}$ or $\underset{\rightarrow}{AB}$.

In *FLAP* we usually denote the magnitude of a vector $\boldsymbol{a}$ by $|\boldsymbol{a}|$, while in your written work you should use $|\underset{\sim}{a}|$. (It is important that you remember to include the underline to denote a vector, and the vertical lines for a magnitude, in your written work.) (☞)

You may understand what you mean, but the reader needs to understand it too!

You will also encounter an alternative notation, used by many authors, in which the magnitude of the vector $\boldsymbol{a}$ is denoted by an unemboldened a.

◆ Which of the following are vector quantities?

(a) Your weight, (b) the pressure at a certain depth in the sea, (c) a charging rhinoceros.

◇ (a) Your weight is a force, and is therefore a vector quantity. (Your mass, which is what you try to reduce when you diet, is a scalar.)

(b) The pressure in a liquid doesn't have a direction, and is therefore a scalar.

(c) A charging rhinoceros certainly has magnitude and direction, but it is not a vector; the resultant of two rhinos, one heading north and another east, is not $\sqrt{2}$ rhinos heading north-east. □ (☞)

The serious point to be made here is that vector quantities (which can be adequately represented by geometric vectors) must add according to the triangle rule.

◆ Is it generally true that $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}|+|\boldsymbol{b}|$?

◇ No, it is not generally true, as we can see from Figure 2a. The length $|\boldsymbol{a}+\boldsymbol{b}|$ of one side of the triangle is not generally equal to the sum of the lengths of the other two sides. □

2.2 Scaling a vector, unit vectors and the zero vector

In order to **scale** a vector $\boldsymbol{a}$ by a factor 2 say, we simply double its length, and denote the result by $2\boldsymbol{a}$, which is compatible with the triangle rule of addition because (as we would expect) we then have $\boldsymbol{a}+\boldsymbol{a}=2\boldsymbol{a}$.

The scaling factor may be positive or negative, so that, for example, if $\boldsymbol{s}$ is a displacement of 10 km north, then $3\boldsymbol{s}$ is a displacement of 30 km north; $-2\boldsymbol{s}$ is a displacement of 20 km south, and dividing by 2, we have $\frac{1}{2}\boldsymbol{s}$, or $\boldsymbol{s}/2$, a displacement of 5 km north.

Unit vectors

If $\boldsymbol{s}$ is a displacement of 10 km north, then dividing $\boldsymbol{s}$ by 10 km produces a vector of magnitude 1 in the northerly direction.

In general, if we divide an arbitrary vector $\boldsymbol{a}$ by its magnitude we obtain a (dimensionless) vector in the same direction as $\boldsymbol{a}$ and of magnitude 1 (a dimensionless number). Such a vector is known as a **unit vector** in the direction of the vector $\boldsymbol{a}$, and is denoted by $\hat{\boldsymbol{a}}$, so that

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$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \qquad (1)$$

◆ If $\hat{\boldsymbol{n}}$ is a unit vector pointing north, and $\hat{\boldsymbol{w}}$ is a unit vector pointing west, what is the magnitude of $\hat{\boldsymbol{n}} + \hat{\boldsymbol{w}}$?

◇ Adding according to the triangle rule, the magnitude of $\hat{\boldsymbol{n}} + \hat{\boldsymbol{w}}$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$, from Pythagoras's theorem. (Notice that it must be the positive square root, since magnitudes are always positive; and there are no dimensions.) □

Question T1 If $\boldsymbol{F}$ represents a force directed vertically downwards, find a unit vector directed vertically upwards. □

You will find the answers to all *Text questions* in Section 6.

The zero vector

We are intent on establishing an algebra of vectors, so clearly one of our requirements is a **zero vector**, $\mathbf{0}$, which is defined so that

$$\boldsymbol{a} + (-1)\boldsymbol{a} = \mathbf{0} \qquad (2)$$

for any vector $\boldsymbol{a}$. The zero vector has magnitude zero, and its direction is undefined (since it is irrelevant). You may have noticed that we have written zero as a vector, since the sum of two vectors must of course be a vector, and you should use a wavy underline with every occurrence of the zero vector in your written work.

zero vector

parallelogram rule

With the addition and subtraction of vectors, it is important that the vectors are of the same physical type, e.g. both forces.

2.3 Addition and subtraction of vectors

The result of adding two vector quantities, say $\boldsymbol{a}$ and $\boldsymbol{b}$, is determined by the triangle rule of Figure 2a, or equivalently by the **parallelogram rule** illustrated in Figure 3a and is of the same physical type as the original vectors. (The choice of rule is merely a matter of taste.)

We follow the practice of ordinary algebra and usually replace $(-1)\boldsymbol{a}$ by $-\boldsymbol{a}$. The vector $-\boldsymbol{a}$ has the same magnitude as the vector $\boldsymbol{a}$, but it acts in the opposite direction (it is *antiparallel* to $\boldsymbol{a}$), for example, in Figure 3b $\overrightarrow{JN} = -\overrightarrow{NJ}$. One further point, which we take for granted in ordinary algebra, is that the order in which we add vectors is immaterial, so

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \qquad (3)$$

and

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} \qquad (4)$$

Note A mathematician would recognize Equations 3 and 4 as the *commutative* and *associative* properties of *vector addition*.

Vector subtraction is achieved by scaling a vector by -1 and then using vector addition as before. For example, in Figure 3b $\overrightarrow{OS} = \boldsymbol{b}$ and therefore

$$\overrightarrow{SO} = -\boldsymbol{b}, \text{ then } \overrightarrow{SP} = \overrightarrow{OP} + \overrightarrow{SO} = \boldsymbol{a} + (-\boldsymbol{b}) = \boldsymbol{a} - \boldsymbol{b}$$

◆ With reference to Figure 3:

(a) Express the vector $\overrightarrow{JW}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) What is the final position of an object which is placed at R and then displaced by the vector $3\boldsymbol{a} - 2\boldsymbol{b}$?

(c) Write down two vectors equal to $2\boldsymbol{a} + 2\boldsymbol{b}$.

◇ (a) $\overrightarrow{JW} = \overrightarrow{JN} + \overrightarrow{NR} + \overrightarrow{RV} + \overrightarrow{VW} = \boldsymbol{b} + \boldsymbol{b} + \boldsymbol{b} + \boldsymbol{a} = 3\boldsymbol{b} + \boldsymbol{a}$.

(b) $3\boldsymbol{a} - 2\boldsymbol{b} = \overrightarrow{RM}$ and so the final position of the object is M.

(c) $\overrightarrow{NX} = \overrightarrow{KU} = 2\boldsymbol{a} + 2\boldsymbol{b}$.

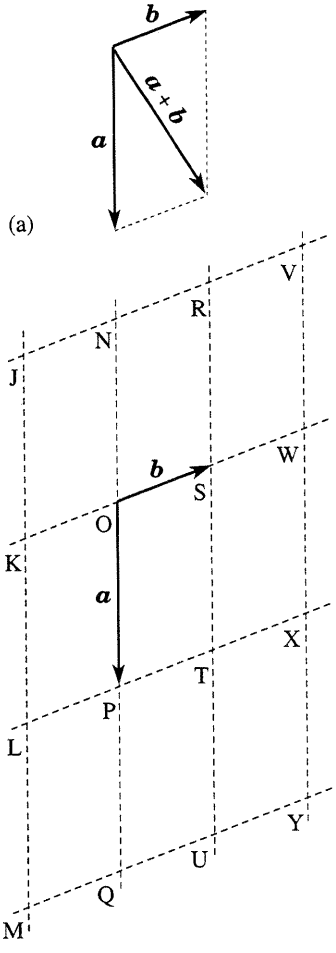


Figure 3 (a) The parallelogram rule of addition. (b) An array of points defining many different vectors.

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Remember that the position in which a geometric vector is drawn is not relevant, and the vectors $\overrightarrow{NX}$ and $\overrightarrow{KU}$ in Figure 3 are equal, in spite of the fact that they are defined by distinct line segments (and the vectors $\overrightarrow{JT}$ and $\overrightarrow{OY}$ would do equally well). Notice also $2\boldsymbol{a} + 2\boldsymbol{b} = 2(\boldsymbol{a} + \boldsymbol{b})$, which is a particular case of a more general rule, which states that for any scalar k

component vectors
resolution

$$k\boldsymbol{a} + k\boldsymbol{b} = k(\boldsymbol{a} + \boldsymbol{b}) \qquad (5)$$

A mathematician would recognize this as the *distributive rule of scalar multiplication* over vector addition.

Question T2 With reference to Figure 3:

(a) Express the vectors $\overrightarrow{KU}$, $\overrightarrow{WL}$ and $\overrightarrow{YK}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Write down two vectors equal to $2\boldsymbol{a} - 3\boldsymbol{b}$. □

2.4 Resolution of vectors, and component vectors

Often when dealing with a vector quantity, we need to know how much the vector contributes in a given direction. For example, consider the two-dimensional situation in which a boat moves with velocity $\boldsymbol{V}$ across a river at an angle θ to the bank, as shown in Figure 4a. To find how long the boat takes to cross the river we need to find the effective velocity normal to (i.e. perpendicular to) the river bank. This can be obtained by treating the vector $\boldsymbol{V}$ as the vector sum of two orthogonal **component vectors** normal to the bank and parallel to it. (☞)

Orthogonal means 'mutually perpendicular'.

Alternatively described as the *vector components* of $\boldsymbol{V}$ by some authors.

If $\hat{\boldsymbol{v}}_n$ and $\hat{\boldsymbol{v}}_p$ are unit vectors normal to, and parallel to the bank, respectively, then $\boldsymbol{V}$ can be written as the sum of two component vectors so that

$$\boldsymbol{V} = |\boldsymbol{V}|\sin\theta\,\hat{\boldsymbol{v}}_n + |\boldsymbol{V}|\cos\theta\,\hat{\boldsymbol{v}}_p$$

The component vectors of $\boldsymbol{V}$ are in this case (see Figure 4b): $|\boldsymbol{V}|\sin\theta\,\hat{\boldsymbol{v}}_n$ in a direction normal to the bank, and of magnitude $|\boldsymbol{V}|\sin\theta$, and, $|\boldsymbol{V}|\cos\theta\,\hat{\boldsymbol{v}}_p$ in a direction parallel to the bank, and of magnitude $|\boldsymbol{V}|\cos\theta$.

If the width of the river is d, we can now see that the time taken for the boat to cross is $d/(|\boldsymbol{V}|\sin\theta)$.

This process of **resolution** (i.e. splitting) into *component vectors* can be thought of as the reverse of vector addition; instead of adding two (or more) vectors to produce a single resultant vector, we are replacing a single vector by the sum of two orthogonal *component vectors*. Note that when these component vectors are added together we obtain our original vector.

In three dimensions we might have three orthogonal component vectors.

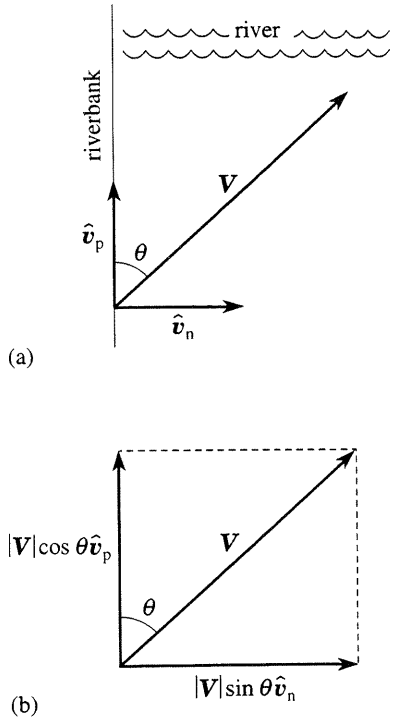


Figure 4 (a) Velocity $\boldsymbol{V}$ at an angle θ to the river bank, with unit vectors $\hat{\boldsymbol{v}}_n$ and $\hat{\boldsymbol{v}}_p$.
(b) Velocity $\boldsymbol{V}$ resolved into component velocities $|\boldsymbol{V}|\sin\theta\,\hat{\boldsymbol{v}}_n$ and $|\boldsymbol{V}|\cos\theta\,\hat{\boldsymbol{v}}_p$ normal to, and parallel to, the river bank.

◆ The string of a simple pendulum exerts a force $\boldsymbol{F}$ of magnitude 15 N on the pendulum bob, at an angle of 15° to the (upward) vertical. Resolve this force into horizontal and vertical component vectors.

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◇ If $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{h}}$ are unit vectors in the (upward) vertical direction and horizontal direction, respectively, then the force in the horizontal direction is

$$|\boldsymbol{F}|\sin\theta\,\hat{\boldsymbol{h}} = (15\sin 15°)\hat{\boldsymbol{h}} = (3.88\,\text{N})\hat{\boldsymbol{h}}$$

and the force in the vertical direction is

$$|\boldsymbol{F}|\cos\theta\,\hat{\boldsymbol{v}} = (15\cos 15°)\hat{\boldsymbol{v}} = (14.5\,\text{N})\hat{\boldsymbol{v}}$$ □

Question T3 A block of wood lies on a rough plane which is inclined at an angle of 25° to the horizontal. It is prevented from sliding down the plane by a static frictional force of magnitude 20 N acting up the plane. Find the horizontal and vertical component vectors of that frictional force. □

3 Vectors in a Cartesian coordinate system

In the previous section you saw how it was possible to represent the velocity of a boat in terms of the unit vectors $\hat{\boldsymbol{v}}_n$ and $\hat{\boldsymbol{v}}_p$ normal to, and parallel to, the bank of a river. This same idea can be extended to three dimensions, and provides a very convenient means of specifying three-dimensional vectors.

3.1 Cartesian unit vectors

Cartesian unit vectors

The use of unit vectors is of particular importance in specifying the positive directions of the axes in a Cartesian coordinate system. In two dimensions the **Cartesian unit vectors** $\boldsymbol{i}$ and $\boldsymbol{j}$ are used to specify the directions of the x- and y-axes as shown in Figure 5.

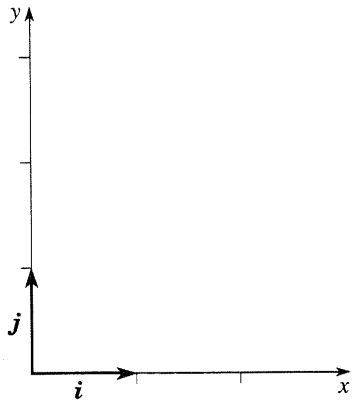


Figure 5 Unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are directed along the x- and y-axes.

In three dimensions a third Cartesian unit vector $\boldsymbol{k}$ is added to specify the direction of the z-axis. However, there are two possible directions, opposite to one another, in which the z-axis can be directed so that it is normal to both the x- and y-axes. To remove any ambiguity about the choice of direction of the z-axis a *right-handed* Cartesian coordinate system is almost invariably used. There are various ways of describing this system, but one simple method only is included here. If the thumb and first two fingers of your *right* hand are arranged approximately mutually perpendicular as in Figure 6, then, if the first and second finger point along the x- and y-axes, respectively, the thumb points along the z-axis in a right-handed system.

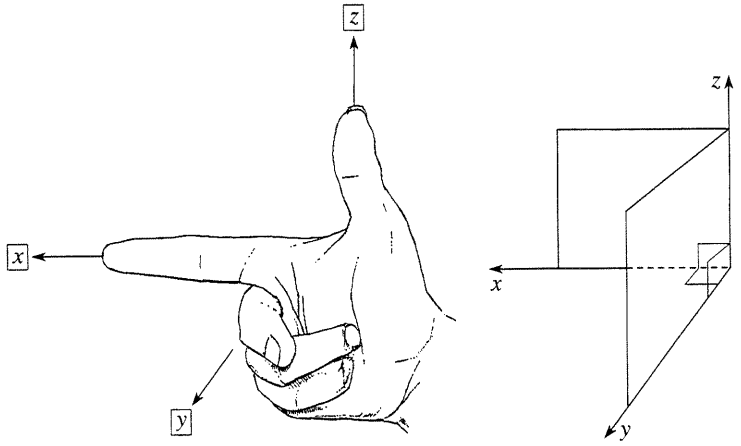


Figure 6 Directions of the axes in a right-handed Cartesian coordinate system.

◆ Which of the following is a right-handed coordinate system:
(a) $\boldsymbol{i}$ points west, $\boldsymbol{j}$ points north and $\boldsymbol{k}$ points vertically upward,
(b) $\boldsymbol{i}$ is vertically downward, $\boldsymbol{j}$ points south and $\boldsymbol{k}$ points west?

◇ Case (b) is right-handed, case (a) is not. □ (☞)

If you have difficulty confirming this answer, try drawing some simple diagrams of cases (a) and (b), and compare them with Figure 6.

Now let us consider how these Cartesian unit vectors may be used to specify the Cartesian component vectors of any vector $\boldsymbol{a}$ in the (x, y) plane, i.e. an arbitrary vector in two dimensions. The vector $\boldsymbol{a}$ can be resolved into two orthogonal component vectors parallel to the x- and y-axes, respectively, as shown in Figure 7.

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If a_x and a_y are the coordinates of the point A, then the component vectors will have magnitudes a_x and a_y. Hence the component vectors will be $a_x\boldsymbol{i}$ and $a_y\boldsymbol{j}$. From vector addition, $\boldsymbol{a}$ can be expressed as:

Cartesian scalar components
Cartesian component vectors
Cartesian form

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} \qquad (6)$$

The important point to notice about Equation 6 is that, once we have specified the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$ the vector $\boldsymbol{a}$ is completely determined (both in magnitude and direction) by the pair of scalar quantities a_x and a_y.

The values a_x and a_y are known as the **Cartesian scalar components** of $\boldsymbol{a}$ (or simply the components of $\boldsymbol{a}$), $a_x\boldsymbol{i}$ and $a_y\boldsymbol{j}$ are the **Cartesian component vectors** of $\boldsymbol{a}$ (or simply the component vectors of $\boldsymbol{a}$).

Some authors refer loosely to just the 'components' of a vector, meaning either the *Cartesian scalar components* or the *Cartesian component vectors*. The context usually makes the meaning clear.

We obtain the full benefit from this method of representing vectors when the idea is extended to three dimensions. In Figure 8 we show how a vector $\boldsymbol{a}$ in three dimensions can be resolved into three orthogonal component vectors parallel to the x-, y- and z-axes. If a_x, a_y and a_z are the coordinates of the point A, then

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \qquad (7)$$

In this case the Cartesian component vectors of $\boldsymbol{a}$ are

$a_x\boldsymbol{i}$, $a_y\boldsymbol{j}$ and $a_z\boldsymbol{k}$

while the Cartesian scalar components of $\boldsymbol{a}$ are

a_x, a_y and a_z

and when we write

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

we say that $\boldsymbol{a}$ is expressed in **Cartesian form.**

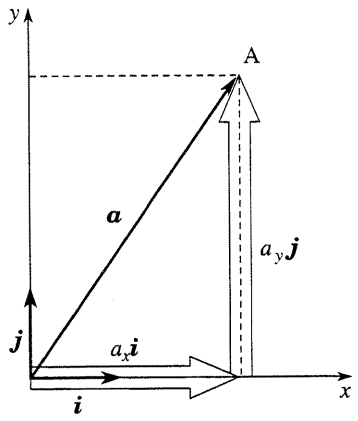


Figure 7 The component vectors of vector $\boldsymbol{a}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

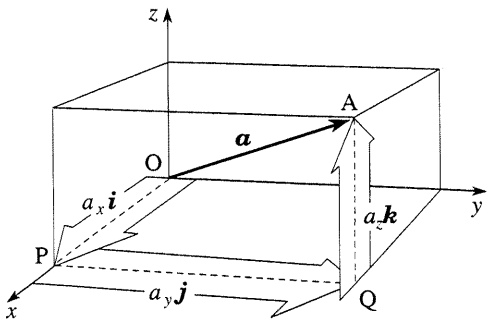


Figure 8 The Cartesian component vectors of a vector $\boldsymbol{a}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

The suffix notation used in the above definitions is quite convenient, and it is natural to express any given vector, $\boldsymbol{r}$ say, in the same form. Thus $r_x\boldsymbol{i}$ is the natural choice of notation for the vector component in the x-direction of the vector $\boldsymbol{r}$, while $r_y\boldsymbol{j}$ and $r_z\boldsymbol{k}$ denote its y- and z-component vectors.

Most of us have great difficulty visualizing objects, and their relative distances from each other, in three dimensions; vector methods largely overcome these problems.

Some authors use the term 'x-component' for both the vector component in the x-direction and the scalar component in the x-direction. In *FLAP* it means the scalar component.

◆ A portable rectangular cabin is 3 m wide, 5 m long and 2.5 m high, and it is placed on a horizontal foundation with its long axis pointing north/south. An origin is chosen in the bottom south-west corner of the cabin, with $\boldsymbol{i}$ pointing east, $\boldsymbol{j}$ pointing north, and $\boldsymbol{k}$ pointing vertically upward.

What vector displacement $\boldsymbol{r}$ is required to move an object from the origin to a point at the centre of the ceiling?

◇ The vector $\boldsymbol{a} = (3\boldsymbol{i} + 5\boldsymbol{j})$ m is the displacement from the origin to the opposite corner of the floor of the cabin. It follows that $\frac{1}{2}\boldsymbol{a} = (1.5\boldsymbol{i} + 2.5\boldsymbol{j})$ m is the displacement from the origin to the centre of the floor. The centre of the ceiling is 2.5 m above this point, and so its displacement from the origin is given by $\boldsymbol{r} = (1.5\boldsymbol{i} + 2.5\boldsymbol{j} + 2.5\boldsymbol{k})$ m. □

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Once we have found the vector displacement $\boldsymbol{r}$ from the origin to the centre of the ceiling, then it is possible to find displacements *from this new point*. For example, the vector displacement of the opposite corner of the floor of the cabin from the origin is $\boldsymbol{a} = 3\boldsymbol{i} + 5\boldsymbol{j}$, and therefore the displacement of the opposite corner of the cabin from the centre of the ceiling is (☞)

$$-\boldsymbol{r} + \boldsymbol{a} = -(1.5\boldsymbol{i} + 2.5\boldsymbol{j} + 2.5\boldsymbol{k})\,\text{m} + (3\boldsymbol{i} + 5\boldsymbol{j})\,\text{m} = (1.5\boldsymbol{i} + 2.5\boldsymbol{j} - 2.5\boldsymbol{k})\,\text{m}$$

We go from the centre of the ceiling to the origin, i.e. $-\boldsymbol{r}$, then from the origin to the far corner of the floor, i.e. $\boldsymbol{a}$.

Question T4 Is the suggested coordinate system for the cabin described above right-handed? What vector displacement $\boldsymbol{s}$ will move an object from the centre of the ceiling to the mid-point of the northern wall? □

3.2 The magnitude and combination of vectors in Cartesian form

We can regard vectors in two dimensions as merely a special case of vectors in three dimensions; their z-component just happens to be zero. So hereafter our results apply equally well to vectors in two or three dimensions.

It is convenient to be able to determine the magnitude of a vector in terms of its Cartesian components, and this may be easily done if we apply Pythagoras's theorem to vector $\boldsymbol{a}$ in Figure 8. First we consider the (x, y) plane only. Triangle OPQ is a right-angled triangle and so:

$$OQ^2 = OP^2 + PQ^2$$

But $OP = a_x$ and $PQ = a_y$, and so:

$$OQ^2 = a_x^2 + a_y^2$$

Now we consider the right-angled triangle OQA:

$$OA^2 = OQ^2 + QA^2$$

But $OQ^2 = a_x^2 + a_y^2$ and $QA^2 = a_z^2$

and so: $OA^2 = a_x^2 + a_y^2 + a_z^2$

But $OA = |\boldsymbol{a}|$

therefore:

$$|\boldsymbol{a}| = (a_x^2 + a_y^2 + a_z^2)^{1/2} \qquad (8)$$

◆ Given that $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ calculate $|\boldsymbol{a}|$.

◇ $|\boldsymbol{a}| = (2^2 + 3^2 + 4^2)^{1/2} = \sqrt{29}$. □

Notice that the sum of squares of real numbers must be positive, and that we always take the positive square root.

Question T5 In Question T4 you were asked to determine a vector displacement $\boldsymbol{s}$. Now determine the magnitude of that displacement. □

As we saw in Subsection 2.2, the scaling of a vector is simply multiplication of the vector by a scalar quantity, and this operation is very straightforward when a vector is expressed in Cartesian form. For example let α be a scalar quantity and let $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, then:

α is the Greek letter alpha.

$$\alpha\boldsymbol{a} = \alpha a_x\boldsymbol{i} + \alpha a_y\boldsymbol{j} + \alpha a_z\boldsymbol{k} \qquad (9)$$

The sum of two vectors in Cartesian form is equally easy.

Let $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$ and $\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$, then:

$$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \qquad (10)$$

and similarly for subtraction

$$\boldsymbol{a} - \boldsymbol{b} = (a_x - b_x)\boldsymbol{i} + (a_y - b_y)\boldsymbol{j} + (a_z - b_z)\boldsymbol{k} \qquad (11)$$

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◆ At 3a.m. a hedgehog of mass 1.5 kg starts crossing the M1 from west to east at a speed of 2.0 km h^{-1}. What is the momentum of the hedgehog?

◇ Taking $\boldsymbol{i}$ to be a unit vector pointing east, the momentum is given by $\boldsymbol{h} = (1.5\text{ kg}) \times (2.0\text{ km h}^{-1})\boldsymbol{i} = 3.0\boldsymbol{i}\text{ kg km h}^{-1}$. □

Momentum is defined as the product of the mass of an object, a scalar, with the velocity of the object, a vector (or, more accurately, the velocity of the centre of mass of the object).

◆ At 3.05a.m. a truck of mass 15 000 kg and speed 50 km h^{-1} passes the same spot travelling north. What is the momentum of the truck?

◇ Taking $\boldsymbol{j}$ to be a unit vector pointing north, the momentum of the truck is $\boldsymbol{t} = (15\,000\text{ kg}) \times (50\text{ km h}^{-1})\boldsymbol{j} = 75 \times 10^4\boldsymbol{j}\text{ kg km h}^{-1}$. □

Question T6 What is the magnitude of the combined momentum of the truck and the hedgehog at 3.06a.m., (and what was the name of the hedgehog)? □

Having considered the operations of scaling, addition and subtraction separately, we are now in a position to combine these processes when using the Cartesian component approach.

Suppose, for example, that $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ then

$$4\boldsymbol{a} - 3\boldsymbol{b} = 4(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) - 3(3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) = -\boldsymbol{i} + 15\boldsymbol{j} + 4\boldsymbol{k}$$

and more generally, for scalars α and β,

$$\alpha\boldsymbol{a} + \beta\boldsymbol{b} = (\alpha a_x + \beta b_x)\boldsymbol{i} + (\alpha a_y + \beta b_y)\boldsymbol{j} + (\alpha a_z + \beta b_z)\boldsymbol{k} \qquad (12)$$

◆ Is there any physical restriction on α, β, $\boldsymbol{a}$ and $\boldsymbol{b}$ in order that Equation 12 makes sense?

◇ Yes, the dimensions of $\alpha\,\boldsymbol{a}$ and $\beta\,\boldsymbol{b}$ must be the same. □

Notice the greater convenience of the Cartesian component approach in comparison with the geometric vector approach in carrying out these operations on vectors.

3.3 An application of vectors in Cartesian form

To complete Section 3 let us consider how the concepts covered in this section may be applied in a specific example.

◆ A particle travels at velocity $\boldsymbol{v}_1$ for 3 s and then velocity $\boldsymbol{v}_2$ for 2 s, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are given by: $\boldsymbol{v}_1 = (2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})\text{ m s}^{-1}$ and $\boldsymbol{v}_2 = (4\boldsymbol{i} + 12\boldsymbol{j} - 3\boldsymbol{k})\text{ m s}^{-1}$. Determine:

(a) the magnitudes of the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$;

(b) the displacements $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ over the first and second intervals;

(c) the distances associated with these displacements;

(d) the total displacement, $\boldsymbol{s}_t$;

(e) the magnitude of the total displacement; and

(f) the unit vector $\hat{\boldsymbol{s}}_t$ in the direction of $\boldsymbol{s}_t$.

Note that we select the positive square root.

◇ (a) To determine the magnitudes of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ we apply Equation 8 which we deduced from Pythagoras's theorem. For $\boldsymbol{v}_1$, the components in m s^{-1} are 2, –3 and 6, and therefore:

$$|\boldsymbol{v}_1| = [2^2 + (-3)^2 + 6^2]^{1/2}\text{ m s}^{-1} = 49^{1/2}\text{ m s}^{-1} = 7\text{ m s}^{-1}$$

Similarly:

$$|\boldsymbol{v}_2| = [4^2 + 12^2 + (-3)^2]^{1/2}\text{ m s}^{-1} = 169^{1/2}\text{ m s}^{-1} = 13\text{ m s}^{-1}$$

(b) The displacement $\boldsymbol{s}$ corresponding to a constant velocity $\boldsymbol{v}$ for a time t is given by $\boldsymbol{s} = t\boldsymbol{v}$, so we can find $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ by scaling the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by the appropriate times. Hence:

$$\boldsymbol{s}_1 = (3\text{ s})\boldsymbol{v}_1 = (6\boldsymbol{i} - 9\boldsymbol{j} + 18\boldsymbol{k})\text{ m}$$

and $$\boldsymbol{s}_2 = (2\text{ s})\boldsymbol{v}_2 = (8\boldsymbol{i} + 24\boldsymbol{j} - 6\boldsymbol{k})\text{ m}$$

(c) The required distances are simply the magnitudes of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. The direct approach to the determination of $|\boldsymbol{s}_1|$ and $|\boldsymbol{s}_2|$ would appear to involve the application of Equation 8 to the vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ found in part (b). However, it

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is easier to make use of the answers obtained for $|\boldsymbol{v}_1|$ and $|\boldsymbol{v}_2|$ in part (a), since $|\boldsymbol{s}_1| = (3\text{ s})|\boldsymbol{v}_1|$ and $|\boldsymbol{s}_2| = (2\text{ s})|\boldsymbol{v}_2|$. Therefore:

ordered triple
ordered pair

$$|\boldsymbol{s}_1| = 3\text{ s} \times 7\text{ m s}^{-1} = 21\text{ m}$$

and $$|\boldsymbol{s}_2| = 2\text{ s} \times 13\text{ m s}^{-1} = 26\text{ m}$$

(d) The total displacement is simply the vector sum $\boldsymbol{s}_t = \boldsymbol{s}_1 + \boldsymbol{s}_2$, where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, are the answers obtained for part (b). Following the Cartesian component vector method for addition, we obtain:

$$\boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{s}_t = [(6+8)\boldsymbol{i} + (-9+24)\boldsymbol{j} + (18-6)\boldsymbol{k}]\text{ m}$$

and therefore $$\boldsymbol{s}_t = [14\,\boldsymbol{i} + 15\boldsymbol{j} + 12\,\boldsymbol{k}]\text{ m}$$

(e) The value of $|\boldsymbol{s}_t|$ can be obtained using Equation 8:

$$|\boldsymbol{s}_t| = [14^2 + 15^2 + 12^2]^{1/2}\text{ m} = \sqrt{565}\text{ m} \approx 23.8\text{ m}$$

(f) The required unit vector is obtained using Equation 1

$$\hat{\boldsymbol{s}}_t = \frac{\boldsymbol{s}_t}{|\boldsymbol{s}_t|} = \frac{14\boldsymbol{i} + 15\boldsymbol{j} + 12\boldsymbol{k}}{\sqrt{565}}$$

therefore $$\hat{\boldsymbol{s}}_t = 0.589\boldsymbol{i} + 0.631\boldsymbol{j} + 0.505\boldsymbol{k}$$ □

You should use Equation 8 to check that the magnitude of this vector is indeed unity.

Question T7 Particle A, of mass 2 kg, travelling with velocity $\boldsymbol{v}_A = (\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})\text{ m s}^{-1}$ collides with particle B, of mass 3 kg, travelling with velocity $\boldsymbol{v}_B = (3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})\text{ m s}^{-1}$. The particles stick together on collision and then move as one combined body with momentum the same as the total momentum of A and B before the collision. Determine:

'momenta' is the plural of momentum.

(a) the momenta of A and B before the collision;

(b) the total (resultant) momentum of A and B before the collision;

(c) the magnitudes of the momenta of A and B before the collision;

(d) and the velocity of the combined body after the collision. □

4 Vectors as ordered triples

4.1 Ordered triples and ordered pairs

A vector, $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ say, in Cartesian form, is often abbreviated to three numbers in brackets, (3, −1, 4), in which the order of the numbers is of crucial importance, for otherwise we would not know which number referred to which component. Such a collection of numbers is called an **ordered triple**.

In general we write

$$\boldsymbol{a} = a_x\,\boldsymbol{i} + a_y\boldsymbol{j} + a_z\,\boldsymbol{k} = (a_x, a_y, a_z) \qquad (13)$$

An ordered triple, such as (3, −1, 4), can mean the coordinates of a point in three dimensions or the vector $3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$, but the context will make it clear which is intended. (In fact this dual meaning is an advantage, as you will see in Subsection 4.4.)

◆ Express the vector $2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ as an ordered triple.

◇ $2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} = (2, -1, 3)$ □

In the case of two-dimensional vectors, we refer to **ordered pairs** rather than ordered triples, for example, $\boldsymbol{v} = 2\boldsymbol{i} + 5\boldsymbol{j} = (2, 5)$. Therefore (2, 5) is the ordered pair which represents the vector $\boldsymbol{v}$.

4.2 Manipulation of ordered triples

The extension of the calculations of the last section, for vectors in Cartesian form, to vectors represented as ordered triples is very easy, but we will state the results 'for the record'.

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If $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$ then

$$|\boldsymbol{a}| = (a_x^2 + a_y^2 + a_z^2)^{1/2} \quad \text{(Eqn 8)}$$

$$\alpha\boldsymbol{a} = \alpha a_x\boldsymbol{i} + \alpha a_y\boldsymbol{j} + \alpha a_z\boldsymbol{k} \quad \text{(Eqn 9)}$$

$$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \quad \text{(Eqn 10)}$$

$$\alpha\boldsymbol{a} + \beta\boldsymbol{b} = (\alpha a_x + \beta b_x, \alpha a_y + \beta b_y, \alpha a_z + \beta b_z) \quad \text{(Eqn 12)}$$

These equations are not identical to those of the previous section, but they are so similar in meaning that they do not warrant separate numbers.

Question T8 Find the magnitude of the vector represented by the ordered triple $(5, -2, -1)$. □

◆ Express $2(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$ as an ordered triple and find the ordered triple which represents the sum of $(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ and $(3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$.

◇ $2(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) = 2(2, -1, 3) = (4, -2, 6)$

$(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + (3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) = (1, 2, -1) + (3, -1, 4) = (4, 1, 3)$ □

Question T9 Find the ordered triple that represents the resultant of the vectors $(2, -1, -2)$ and $(3, -2, 7)$, and find also the ordered triple which represents the unit vector in the direction opposite to that of the resultant. □

4.3 An application of ordered triples

The following exercises are intended to illustrate how vectors may be manipulated using the ordered triple notation.

◆ Forces $\boldsymbol{F}_1 = (2, -1, 4)$ N and $\boldsymbol{F}_2 = (1, 3, 1)$ N act simultaneously on a particle. Determine:

(a) the magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$;

(b) the resultant of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$;

(c) the magnitude of this resultant;

(d) the vector required to cancel the combined effect of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$;

(e) the vectors $2\boldsymbol{F}_1$, $5\boldsymbol{F}_2$ and $2\boldsymbol{F}_1 + 5\boldsymbol{F}_2$.

◇ (a) $|\boldsymbol{F}_1| = [2^2 + (-1)^2 + 4^2]^{1/2}$ N $= [4 + 1 + 16]^{1/2}$ N $= 21^{1/2}$ N $= 4.58$ N.

$|\boldsymbol{F}_2| = [1^2 + 3^2 + 1^2]^{1/2}$ N $= [1 + 9 + 1]^{1/2}$ N $= 11^{1/2}$ N $= 3.32$ N

(b) $\boldsymbol{F}_1 + \boldsymbol{F}_2 = (2 + 1, -1 + 3, 4 + 1)$ N $= (3, 2, 5)$ N.

(c) $|\boldsymbol{F}_1 + \boldsymbol{F}_2| = [3^2 + 2^2 + 5^2]^{1/2}$ N $= [9 + 4 + 25]^{1/2}$ N $= 38^{1/2}$ N $= 6.16$ N.

(d) If $\boldsymbol{F}_1 + \boldsymbol{F}_2 = (3, 2, 5)$ N, then the force vector required to cancel the effect of $\boldsymbol{F}_1 + \boldsymbol{F}_2$ is $-(\boldsymbol{F}_1 + \boldsymbol{F}_2) = (-3, -2, -5)$ N.

(e) $2\boldsymbol{F}_1 = 2(2, -1, 4)$ N $= (4, -2, 8)$ N.

$5\boldsymbol{F}_2 = 5(1, 3, 1)$ N $= (5, 15, 5)$ N

so that $2\boldsymbol{F}_1 + 5\boldsymbol{F}_2 = (4 + 5, -2 + 15, 8 + 5)$ N $= (9, 13, 13)$ N. □

Question T10 A particle travels with velocity $\boldsymbol{v}_1 = (2, 1, -2)\,\text{m s}^{-1}$ for 5 s, then with velocity $\boldsymbol{v}_2 = (-1, 2, 3)\,\text{m s}^{-1}$ for 2 s, and finally with velocity $\boldsymbol{v}_3 = (-2, -1, 2)\,\text{m s}^{-1}$ for 3 s. Find the total displacement of the particle over the 10 s, the magnitude of this total displacement, and the unit vector in the direction of the total displacement. □

4.4 Position vectors

There are two main applications of vectors in physics. The first concerns the mathematical modelling (or representation, if you prefer) of physical quantities such as force, displacement and velocity. The second is equally important, and concerns the location of objects in three dimensions, and is essentially a form of geometry. Generally we do not associate a vector with a particular point in space (although later you may encounter applications where it is desirable to do just that), and

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vectors such as $\overrightarrow{JN}$ and $\overrightarrow{UY}$ in Figure 3 are defined to be the same vector. On the other hand, the vector $\boldsymbol{a}$ in Figure 8 specifies the position of the point A provided that we know that the end of the vector is fixed at the origin. Vectors that are used in this way to determine the positions of points are often known as **position vectors**.

position vector

◆ Two points A and B are specified by the position vectors

$$\overrightarrow{OA} = (1, -3, 5)\text{ cm} \quad \text{and} \quad \overrightarrow{OB} = (-2, 2, 4)\text{ cm}$$

Find the distance between the points A and B, and the position vector of the mid-point M of AB.

Is a position vector $\boldsymbol{r}$ identical to a displacement vector $\boldsymbol{r}$?

◇ The displacement vector from A to B is defined by

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = [(-2, 2, 4) - (1, -3, 5)]\text{ cm} = (-3, 5, -1)\text{ cm}$$

so that the distance from A to B is $\overrightarrow{AB} = [(9 + 25 + 1)^{1/2}]\text{ cm} \approx 5.92\text{ cm}$.

The mid-point of AB has coordinates $\frac{1}{2}[(-2, 2, 4) + (1, -3, 5)]$ cm so that the position vector $\overrightarrow{OM} = \frac{1}{2}[(-2, 2, 4) + (1, -3, 5)]\text{ cm} = (-0.5, -0.5, 4.5)\text{ cm}$

The position vector $\boldsymbol{r}$ and the displacement vector $\boldsymbol{r}$ are identical, they are the same vector; but the interpretation of them is different. The fact that we are told that $\boldsymbol{r}$ is a displacement vector tells us only the magnitude and direction of the displacement. We can use that displacement to specify a particular point only if we first specify a point from which to make the displacement. On the other hand, if we are told that $\boldsymbol{r}$ is a position vector we know that the position of the point A is determined if the vector is placed with its tail at the origin. In a sense, position vectors are simply a special class of displacement vectors since position vectors define displacements from the origin.

Notice that in the above exercise we were quite happy to use exactly the same notation for the coordinates of M and for the position vector $\overrightarrow{OM}$. The context makes it absolutely clear which is intended, and, in any case, there is very little difference between saying that a point P is determined by the coordinates (1, 2, 3) and that P is determined by the position vector (1, 2, 3).

Question T11 A particle is moving with velocity $\boldsymbol{v} = (1, 1, 2)\text{ m s}^{-1}$ and at time $t = 0$ it is at the point P with position vector $\overrightarrow{OP} = (2, 3, -4)$ m. What is the position vector $\overrightarrow{OQ}$ of the particle at time $t = 5$ s? □

5 Closing items

5.1 Module summary

1 *Scalar quantities* can be specified completely by a single number together with an appropriate unit of measurement.

2 *Vector quantities* can be specified by a magnitude and a direction. Geometric vectors are represented by arrows or *directed line segments* and vector quantities are often represented pictorially by geometric vectors. In print, vectors are denoted by bold typeface, and in handwritten material, by a wavy underline.

3 The *magnitude* of a vector is a non-negative scalar that represents the 'length' or 'size' of that vector. The magnitude of $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$ (or sometimes by a) in print, and by $|\underset{\sim}{a}|$ in handwritten material.

4 Any vector $\boldsymbol{a}$ may be multiplied by a scalar α to produce a scaled vector $\alpha\boldsymbol{a}$ which points in the same direction as $\boldsymbol{a}$ if $\alpha > 0$ and in the opposite direction if $\alpha < 0$. The magnitude of $\alpha\boldsymbol{a}$ is $|\alpha\boldsymbol{a}| = |\alpha||\boldsymbol{a}|$.

5 If any non-zero vector $\boldsymbol{a}$ is divided by $|\boldsymbol{a}|$, a vector of unit magnitude is obtained which points in the same direction as $\boldsymbol{a}$. Such a vector is called a *unit vector* and is denoted by $\hat{\boldsymbol{a}}$.

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6 Vectors may be added geometrically using either the *triangle* or *parallelogram rule* (see Figures 2 and 3).

7 A vector may be *resolved* into *component vectors* along appropriately chosen directions. Given a vector $\boldsymbol{a}$, its orthogonal *component vectors* parallel and normal to a direction inclined at an angle θ to $\boldsymbol{a}$ are of magnitude

$$|\boldsymbol{a}_p| = |\boldsymbol{a}| \cos\theta \quad \text{and} \quad |\boldsymbol{a}_n| = |\boldsymbol{a}| \sin\theta$$

8 The *Cartesian unit vectors* in the directions of the Cartesian axes x, y and z are denoted by $\boldsymbol{i}$,$\boldsymbol{j}$ and $\boldsymbol{k}$.

9 A vector $\boldsymbol{a}$ can be expressed in Cartesian form as

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k} \qquad \text{(Eqn 7)}$$

The scalars a_x, a_y and a_z are called the Cartesian scalar components of $\boldsymbol{a}$, whereas the vectors $a_x \boldsymbol{i}$, $a_y \boldsymbol{j}$ and $a_z \boldsymbol{k}$ are called the Cartesian component vectors of $\boldsymbol{a}$.

Both are often described loosely as the 'components' of $\boldsymbol{a}$.

10 The *magnitude* of the vector $\boldsymbol{a}$ is given by

$$|\boldsymbol{a}| = (a_x^2 + a_y^2 + a_z^2)^{1/2} \qquad \text{(Eqn 8)}$$

11 The operations of *scaling* and *vector addition* take the following Cartesian algebraic forms (for any scalar α and vectors $\boldsymbol{a}$ and $\boldsymbol{b}$):

$$\alpha \boldsymbol{a} = \alpha a_x \boldsymbol{i} + \alpha a_y \boldsymbol{j} + \alpha a_z \boldsymbol{k}$$

and $$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \qquad \text{(Eqn 10)}$$

12 A vector represented by $\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}$ can also be represented by the abbreviated notation of an *ordered triple*:

$$\boldsymbol{a} = (a_x, a_y, a_z)$$

13 The operations of scaling and vector addition take the following abbreviated forms (for any scalar α and vectors $\boldsymbol{a}$ and $\boldsymbol{b}$):

$$\alpha \boldsymbol{a} = (\alpha a_x, \alpha a_y, \alpha a_z)$$

and $$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

14 Vectors may be used to determine the position of points relative to a chosen origin, and they are then known as *position vectors*.

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Identify quantities as being scalars or vectors, given the definitions of the quantities.

A3 Recognize and use the notations ($\boldsymbol{a}$,$\underset{\sim}{a}$, $|\boldsymbol{a}|$, $|\underset{\sim}{a}|$) to represent vectors and their magnitudes.

A4 Carry out and represent graphically the operations of scaling, addition and subtraction of vectors, and of resolving a vector into orthogonal component vectors.

A5 Determine a unit vector in the same direction as a given vector.

A6 Use Cartesian unit vectors and Cartesian scalar components to represent a given vector in Cartesian form.

A7 Evaluate the magnitude of any vector in terms of its Cartesian components.

A8 Scale, add and subtract vectors in Cartesian form.

A9 Use Cartesian scalar components to represent a given vector as an ordered triple (or an ordered pair in two dimensions).

A10 Scale, add and subtract vectors using the ordered triple notation for vectors.

A11 Use a position vector to specify the location of a point.

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5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2*) The electric field at any point in space can be defined as the electrical force experienced by a positive charge at that point, divided by the magnitude of the charge. Given that charge is a scalar quantity, decide whether electric field is a scalar or a vector quantity, and justify your decision.

Question E2 (*A4*) On Treasure Island, Captain Flint decides to bury his treasure at the point at which he arrives after making a displacement of 20 m west and then a displacement of 10 m north, starting from a distinctive rock which he uses as his reference point. Illustrate these displacements graphically, and find the magnitude and direction of the resultant displacement. Find also the component of the resultant displacement in the north-west direction.

Question E3 (*A3*, *A5*, *A6*, *A7* and *A8*) Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by

$$\boldsymbol{a} = -\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k} \quad \text{and} \quad \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$$

Find $\boldsymbol{a} + \boldsymbol{b}$, $|\boldsymbol{a} + \boldsymbol{b}|$ and the unit vector in the direction of $\boldsymbol{a} + \boldsymbol{b}$.

Question E4 (*A3*, *A6*, *A7* and *A8*) Given the forces: $\boldsymbol{F}_1 = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$ N, and $\boldsymbol{F}_2 = (3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k})$ N, find the force given by $3\boldsymbol{F}_1 + 2\boldsymbol{F}_2$, the magnitude of this force and its Cartesian vector component along the z-axis.

Question E5 (*A3*, *A9* and *A10*) Given the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in Cartesian form as:

$$\boldsymbol{a} = -\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k} \quad \text{and} \quad \boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

express the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} + \boldsymbol{b}$ in terms of ordered triples. What physical restriction must apply to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for the resultant to have any sensible meaning?

Question E6 (*A9* and *A10*) Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by: $\boldsymbol{a} = (2, 1, -1)$ and $\boldsymbol{b} = (1, -3, 2)$. Find, in ordered triple notation, the vectors $2\boldsymbol{a} + 5\boldsymbol{b}$, $3\boldsymbol{a} - 2\boldsymbol{b}$, and the vector $\boldsymbol{c}$ such that $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = (0, 0, 0)$.

Question E7 (*A9* and *A10*) Given the forces $\boldsymbol{F}_1 = (2, 0, 2)$ N, $\boldsymbol{F}_2 = (1, -2, 3)$ N and $\boldsymbol{F}_3 = (-3, 1, 2)$ N, determine:

(a) the ordered triple representing $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$;

(b) the magnitude of $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$;

(c) the ordered triple representing the unit vector in the direction of $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$;

(d) the component vector in the z-direction of the force which would completely counteract the force $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$;

(e) the scalar component of $\boldsymbol{F}_1 + \boldsymbol{F}_2 - \boldsymbol{F}_3$ in the y-direction.

Question E8 (*A11*) A pyramid has a square base of length 100 m and its height is 20 m. Cartesian coordinates are chosen with one corner of the base as the origin, with $\boldsymbol{i}$ and $\boldsymbol{j}$ in the direction of the adjacent edges of the base, and with $\boldsymbol{k}$ vertically upward. Find the position vector of the vertex of the pyramid.

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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6 Answers and comments

Fast track answers

F1 (a) If the particle is in equilibrium then $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$ and therefore

$$\boldsymbol{F}_3 = -(\boldsymbol{F}_1 + \boldsymbol{F}_2)$$

so that $\boldsymbol{F}_3 = -[(-\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) + (-\boldsymbol{i} - 5\boldsymbol{k})]$ N

$$= -(-2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})\,\text{N} = (2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})\,\text{N}$$

(b) Now $2\boldsymbol{F}_1 = (-2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k})$ N, $\boldsymbol{F}_2 = (-\boldsymbol{i} - 5\boldsymbol{k})$ N

and $3\boldsymbol{F}_3 = (6\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k})$ N

so that $2\boldsymbol{F}_1 + \boldsymbol{F}_2 + 3\boldsymbol{F}_3 = (3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k})$ N

Using Equation 8,

$$|2\boldsymbol{F}_1 + \boldsymbol{F}_2 + 3\boldsymbol{F}_3| = [3^2 + (-2)^2 + 6^2]^{1/2}\,\text{N} = 7\,\text{N}$$

Using Equation 1, the required unit vector is obtained by dividing the vector $2\boldsymbol{F}_1 + \boldsymbol{F}_2 + 3\boldsymbol{F}_3$ by $|2\boldsymbol{F}_1 + \boldsymbol{F}_2 + 3\boldsymbol{F}_3|$, it is therefore

$$\left(\frac{3}{7}\boldsymbol{i} - \frac{2}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k}\right) = 0.429\boldsymbol{i} - 0.286\boldsymbol{j} + 0.857\boldsymbol{k}$$

F2 $\boldsymbol{v}_p + \boldsymbol{v}_c = (2, 2, -1)\,\text{m s}^{-1} + (-1, 1, 3)\,\text{m s}^{-1} = (1, 3, 2)\,\text{m s}^{-1}$

therefore $|\boldsymbol{v}_p + \boldsymbol{v}_c| = [1^2 + 3^2 + 2^2]^{1/2}\,\text{m s}^{-1} = \sqrt{14}\,\text{m s}^{-1}$

therefore time = distance/speed = $(5/\sqrt{14})$ s = 1.34 s

Ready to study answers

R1 From the definition of the tangent of an angle:

$$\tan \text{B}\hat{\text{A}}\text{C} \approx \left(\frac{5}{12}\right) \text{ so that } \text{B}\hat{\text{A}}\text{C} = \arctan\left(\frac{5}{12}\right) \approx 22.6°$$

so $\text{A}\hat{\text{C}}\text{B} \approx 90° - 22.6° \approx 67.4°$

From *Pythagoras's theorem*: AC = $(25 + 144)^{1/2}$ cm = 13 cm

Consult *Pythagoras's theorem* and *tangent* in the *Glossary* for further information.

R2 Let the angle of inclination be θ (i.e. the angle between the ladder and the ground), then from the definition of the *sine of an angle*:

$\sin\theta = 4.5/5$ so that $\theta = \arcsin(4.5/5) \approx 64.2°$

From the definition of *cosine*, the distance of the foot of the ladder from the wall is $(5\cos\theta)$ m = $(5\cos 64.2°)$ m ≈ 2.18 m

Consult *cosine* and *sine* of an angle in the *Glossary* for further information.

R3 You may find it helpful to plot A and B on a *Cartesian coordinate system.*

From *Pythagoras's theorem*: OA = $(1^2 + 2^2)^{1/2} = 5^{1/2} \approx 2.24$

Similarly: OB = $(4^2 + 3^2)^{1/2} = 25^{1/2} = 5$

Also: AB = $[(4 - 1)^2 + (3 - 2)^2]^{1/2} = 10^{1/2} \approx 3.16$

The angle between OA and the x-axis is arctan (2), while the angle between OB and the x-axis is arctan (3/4), so that

$$\text{A}\hat{\text{O}}\text{B} = \arctan(2) - \arctan(3/4) \approx 63.43° - 36.87° \approx 26.6°$$

Text answers

T1 If $\boldsymbol{F}$ is vertically downwards, then the required unit vector upwards = $-\boldsymbol{F}/|\boldsymbol{F}|$.

T2 (a) In Figure 3,

$\overrightarrow{KU} = 2\boldsymbol{a} + 2\boldsymbol{b}$, $\overrightarrow{WL} = \boldsymbol{a} - 3\boldsymbol{b}$ and $\overrightarrow{YK} = -2\boldsymbol{a} - 3\boldsymbol{b}$.

(b) $2\boldsymbol{a} - 3\boldsymbol{b} = \overrightarrow{WM} = \overrightarrow{VL}$.

T3 If $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{h}}$ are unit vectors in the direction of the upward vertical, and a horizontal in the direction of steepest slope, respectively, then the horizontal and vertical component vectors are:

$$[(20\cos 25°)\,\text{N}]\,\hat{\boldsymbol{h}} \approx (18.13\,\text{N})\,\hat{\boldsymbol{h}}$$

and $[(20\sin 25°)\,\text{N}]\,\hat{\boldsymbol{v}} \approx (8.45\,\text{N})\,\hat{\boldsymbol{v}}$

T4 The coordinate system is right-handed. The displacement from the origin to the mid-point of the north wall is $\boldsymbol{p} = (1.5\boldsymbol{i} + 5\boldsymbol{j} + 1.25\boldsymbol{k})$ m. The displacement from the centre of the ceiling to the mid-point of the north wall is given by

$$\boldsymbol{s} = -\boldsymbol{r} + \boldsymbol{p}$$

$$= -(1.5\boldsymbol{i} + 2.5\boldsymbol{j} + 2.5\boldsymbol{k})\,\text{m} + (1.5\boldsymbol{i} + 5\boldsymbol{j} + 1.25\boldsymbol{k})\,\text{m}$$

$$= (2.5\boldsymbol{j} - 1.25\boldsymbol{k})\,\text{m}$$

T5 The magnitude of the displacement is given by Equation 8:

$$|\boldsymbol{s}| = [0^2 + (2.5)^2 + (1.25)^2]^{1/2}\,\text{m} \approx 2.80\,\text{m}$$

T6 The combined momentum is

$$\boldsymbol{h} + \boldsymbol{t} = (3.0\,\boldsymbol{i} + 75 \times 10^4\,\boldsymbol{j})\,\text{kg km h}^{-1}$$

and

$$|\boldsymbol{h} + \boldsymbol{t}| = \sqrt{(3.0)^2 + (75 \times 10^4)^2}\,\text{kg km h}^{-1}$$

$$\approx 75 \times 10^4\,\text{kg km h}^{-1}$$

In the five minutes between 3a.m. and 3.05a.m. the hedgehog covered a distance of about 167 m, and since the total width of the M1 (including both carriageways, the central reservation and the emergency lanes) is about 40 m, the hedgehog was long gone by the time the truck passed. The answer for the momentum is the same whether they collide or not. (The hedgehog's name was Lucky!)

T7 (a) Since momentum = $m\boldsymbol{v}$, the required momenta are obtained by scaling the given velocities by the appropriate masses. Hence:

$$\text{momentum of A} = (2\,\text{kg})\,\boldsymbol{v}_\text{A} = 2(\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})\,\text{kg m s}^{-1}$$

$$= (2\boldsymbol{i} - 4\boldsymbol{j} + 8\boldsymbol{k})\,\text{kg m s}^{-1}$$

and $\text{momentum of B} = (3\,\text{kg})\,\boldsymbol{v}_\text{B} = 3(3\,\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})\,\text{kg m s}^{-1}$

$$= (9\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})\,\text{kg m s}^{-1}$$

(b) The total momentum = $(2\,\text{kg})\,\boldsymbol{v}_\text{A} + (3\,\text{kg})\,\boldsymbol{v}_\text{B}$

$$= [(2\boldsymbol{i} - 4\boldsymbol{j} + 8\boldsymbol{k}) + (9\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})]\,\text{kg m s}^{-1}$$

$$= (11\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\,\text{kg m s}^{-1}$$

(c) The magnitudes of the initial momenta are obtained using Equation 8:

$$|(2\,\text{kg})\,\boldsymbol{v}_\text{A}| = [2^2 + (-4)^2 + 8^2]^{1/2}\,\text{kg m s}^{-1}$$

$$= 84^{1/2}\,\text{kg m s}^{-1} \approx 9.17\,\text{kg m s}^{-1}$$

and $|(3\,\text{kg})\,\boldsymbol{v}_\text{B}| = [9^2 + 3^2 + (-6)^2]^{1/2}\,\text{kg m s}^{-1}$

$$= 126^{1/2}\,\text{kg m s}^{-1} \approx 11.2\,\text{kg m s}^{-1}$$

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(d) The final total momentum equals the total mass of the combined body multiplied by the final velocity, $\boldsymbol{v}_F$, say. Therefore $\boldsymbol{v}_F$ is obtained by dividing the final momentum by the total mass, so that

$$\boldsymbol{v}_F = \left(\frac{11\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}}{5}\right) \text{m s}^{-1} = (2.2\boldsymbol{i} - 0.2\boldsymbol{j} + 0.4\boldsymbol{k}) \text{ m s}^{-1}$$

T8 Equation 8 gives the expression for the magnitude of the vector. Hence:

$$|(5, -2, -1)| = [5^2 + (-2)^2 + (-1)^2]^{1/2} = 30^{1/2} \approx 5.48$$

T9 The resultant of $(2, -1, -2)$ and $(3, -2, 7)$ is given by $(5, -3, 5)$ in ordered triple notation. To find the required unit vector we need to find the magnitude of the resultant, and this is done using Equation 8. Therefore:

$$\text{magnitude} = [5^2 + (-3)^2 + 5^2]^{1/2} = 59^{1/2} = 7.68$$

The required unit vector is oppositely directed to the resultant and can be obtained using Equation 1:

$$\text{required unit vector} = -\frac{(5, -3, 5)}{\sqrt{59}}$$

$$\approx (-0.651, 0.391, -0.651)$$

T10 The individual displacements $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$, say, are obtained by scaling the velocities by the appropriate time intervals. Therefore

$$\boldsymbol{s}_1 = t_1\boldsymbol{v}_1, \quad \boldsymbol{s}_2 = t_2\boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{s}_3 = t_3\boldsymbol{v}_3$$

so that

$$\boldsymbol{s}_1 = 5(2, 1, -2) \text{ m} = (10, 5, -10) \text{ m}$$

$$\boldsymbol{s}_2 = 2(-1, 2, 3) \text{ m} = (-2, 4, 6) \text{ m}$$

and $$\boldsymbol{s}_3 = 3(-2, -1, 2) \text{ m} = (-6, -3, 6) \text{ m}$$

The resultant displacement is obtained by adding $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$. Therefore

$$\boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3 = (2, 6, 2) \text{ m}$$

The magnitude of this resultant is given by Equation 8:

$$|\boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3| = [2^2 + 6^2 + 2^2]^{1/2} \text{ m} = 44^{1/2} \text{ m} \approx 6.63 \text{ m}$$

The unit vector is given by Equation 1:

$$\text{unit vector} = \frac{\boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3}{|\boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3|} = \frac{(2, 6, 2)}{\sqrt{44}}$$

$$\approx (0.302, 0.905, 0.302)$$

T11 The displacement of the particle after 5 s is

$$\overrightarrow{PQ} = (5 \text{ s})\,\boldsymbol{v} = 5(1, 1, 2) \text{ m}$$

so that $$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = (2, 3, -4) \text{ m} + 5(1, 1, 2) \text{ m}$$

$$= (7, 8, 6) \text{ m}$$

Exit test answers

E1 Since electric field is a vector divided by a scalar it must be a vector quantity.

(Reread Subsection 2.1 if you had difficulty with this question.)

E2 See Figure 9. From this diagram the resultant displacement can be seen to be represented by the hypotenuse of the right-angled triangle. Therefore, from Pythagoras's theorem, the magnitude of the resultant displacement is:

$$[20^2 + 10^2]^{1/2} \text{ m} \approx 22.4 \text{ m}$$

Since $\tan\theta = 1/2$ it follows that $\theta = \arctan(1/2) \approx 26.6°$

The component in the north-west direction is given by

$$22.4 \cos(45° - \theta) \text{ m} \approx 22.4 \cos(45 - 26.6°) \text{ m} \approx 21.2 \text{ m}$$

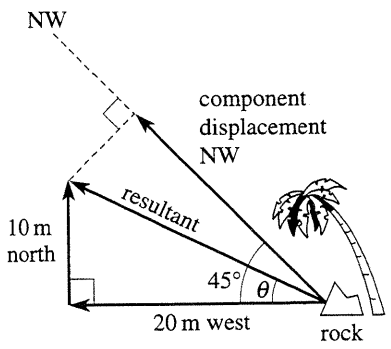


Figure 9 See Answer E2.

(Reread Subsections 2.3 and 2.4 if you had difficulty with this question.)

E3 $\boldsymbol{a} = -\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

and therefore $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$

The magnitude of $\boldsymbol{a} + \boldsymbol{b}$ is given by Equation 8

$$|\boldsymbol{a} + \boldsymbol{b}| = [2^2 + 3^2 + (-6)^2]^{1/2} = \sqrt{49} = 7$$

$$\text{The unit vector} = \frac{\boldsymbol{a} + \boldsymbol{b}}{|\boldsymbol{a} + \boldsymbol{b}|} = \frac{2\boldsymbol{i} + 3\boldsymbol{i} - 6\boldsymbol{k}}{7}$$

$$\approx 0.286\boldsymbol{i} + 0.429\boldsymbol{j} - 0.857\boldsymbol{k}$$

(Reread Subsections 2.2, 3.1 and 3.2 if you had difficulty with this question.)

E4 Given $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, we first evaluate $3\boldsymbol{F}_1$ and $2\boldsymbol{F}_2$ by scaling:

$$3\boldsymbol{F}_1 = 3(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) \text{ N} = (3\boldsymbol{i} - 6\boldsymbol{j} + 9\boldsymbol{k}) \text{ N}$$

and $$2\boldsymbol{F}_2 = 2(3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) \text{ N} = (6\boldsymbol{i} + 2\boldsymbol{j} - 8\boldsymbol{k}) \text{ N}$$

Now we simply add these two vectors:

$$3\boldsymbol{F}_1 + 2\boldsymbol{F}_2 = (9\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) \text{ N}$$

The magnitude of this vector is found from Equation 8:

$$|3\boldsymbol{F}_1 + 2\boldsymbol{F}_2| = [9^2 + (-4)^2 + 1^2]^{1/2} \text{ N} = \sqrt{98} \text{ N} \approx 9.90 \text{ N}$$

The Cartesian component vector along the z-axis is obtained from the expression for the resultant. It is simply the 'k term', which in this case is $\boldsymbol{k}$ N.

(Reread Subsections 3.1and 3.2 if you had difficulty with this question.)

E5 Using the ordered triple notation:

$$\boldsymbol{a} = -\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k} = (-1, 4, -2)$$

and $$\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} = (2, -1, 2)$$

so that $$\boldsymbol{a} + \boldsymbol{b} = (-1, 4, -2) + (2, -1, 2) = (1, 3, 0)$$

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For the resultant to have any physical meaning, the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ must have the same dimensions.

(Reread Subsections 4.1 and 4.2 if you had difficulty with this question.)

E6 To find $2\boldsymbol{a} + 5\boldsymbol{b}$, first we scale vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by 2 and 5, respectively:

$$2\boldsymbol{a} = 2(2, 1, -1) = (4, 2, -2)$$

and $$5\boldsymbol{b} = 5(1, -3, 2) = (5, -15, 10)$$

then $$2\boldsymbol{a} + 5\boldsymbol{b} = (4, 2, -2) + (5, -15, 10) = (9, -13, 8)$$

Similarly $$3\boldsymbol{a} - 2\boldsymbol{b} = (6, 3, -3) - (2, -6, 4) = (4, 9, -7)$$

Since $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$ it follows that

$$\boldsymbol{c} = -(\boldsymbol{a} + \boldsymbol{b}) = -[(2, 1, -1) + (1, -3, 2)] = (-3, 2, -1)$$

(Reread Subsections 4.2, 4.3, and 4.4 if you had difficulty with this question.)

E7 (a) $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = [(2, 0, 2) + (1, -2, 3) + (-3, 1, 2)]$ N $= (0, -1, 7)$ N

(b) The magnitude is obtained from Equation 8:

$$|\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3| = [0^2 + (-1)^2 + 7^2]^{1/2} \text{ N} = \sqrt{50} \text{ N}$$
$$\approx 7.07 \text{ N}$$

(c) The unit vector is $\dfrac{\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3}{|\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3|} = \left(0, -\dfrac{1}{\sqrt{50}}, \dfrac{7}{\sqrt{50}}\right)$

$$\approx (0, -0.141, 0.990)$$

(d) The force which would counteract $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$ is

$$-(\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3) = -(0, -1, 7) \text{ N}$$

and the vector component of this force in the z-direction is $(0, 0, -7)$ N, alternatively written as $-7\boldsymbol{k}$ N.

(e) $\boldsymbol{F}_1 + \boldsymbol{F}_2 - \boldsymbol{F}_3 = [(2, 0, 2) + (1, -2, 3) - (-3, 1, 2)]$ N

$$= (6, -3, 3) \text{ N}$$

Therefore the scalar component in the y-direction is -3 N.

(Reread Subsections 4.2 and 4.3 if you had difficulty with this question.)

E8 The centre of the (square) base is at (50, 50, 0) m and the vertex is 20 m above this point at (50, 50, 20) m.

(Reread Subsection 4.4 if you had difficulty with this question.)

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