Trigonometric functions

Module M1.6

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1 Opening items

1.1 Module introduction

Trigonometric functions have a wide range of application in physics; examples include the addition and resolution of vectors (such as forces), the description of simple harmonic motion and the formulation of quantum theories of the atom. Trigonometric functions are also important for solving certain differential equations, a topic which is considered in some detail elsewhere in *FLAP*.

In Section 2 of this module we begin by looking at the measurement of *angles* in *degrees* and in *radians*. We then discuss some basic ideas about *triangles*, including *Pythagoras's theorem*, and we use *right-angled triangles* to introduce the *trigonometric ratios* (*sin* θ, *cos* θ and *tan* θ) and the *reciprocal trigonometric ratios* (*sec* θ, *cosec* θ and *cot* θ). In Section 3 we extend this discussion to include the *trigonometric functions* ($\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$) and the *reciprocal trigonometric functions* (*cosec* (θ), *sec* (θ) and *cot* (θ)). These *periodic functions* generalize the corresponding ratios since the *argument* θ may take on values that are outside the range 0 to $\pi/2$. Subsection 3.2 discusses the related *inverse trigonometric functions* (*arcsin* (x), *arccos* (x) and *arctan* (x)), paying particular attention to the conditions needed to ensure they are defined. We end, in Section 4, by showing how the sides and angles of any triangle are related by the *sine rule* and the *cosine rule* and by listing some useful *identities* involving trigonometric functions.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 6. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 State Pythagoras's theorem.

Question F2 Define the ratios sin θ, cos θ and tan θ in terms of the sides of the right-angled triangle. What are the exact values (i.e. don't use your calculator) of sin (45°) and tan ($\pi/4$)?

Question F3 Write down the sine and cosine rules for a triangle. Calculate the angles of a triangle which has sides of length 2 m, 2 m and 3 m.

Question F4 Sketch graphs of the functions, cosec (θ) and sec (θ), for $-2\pi \leq \theta \leq 2\pi$ and cot (θ) for $-3\pi/2 < \theta < 3\pi/2$.

Question F5 Sketch graphs of the functions, arccosec (x) and arccot (x) over the range $-10 < x < 10$.

1.3 Ready to study?

Study comment In order to study this module you will need to understand the following terms: *constant*, *decimal places*, *function*, *power*, *reciprocal* and *variable*. An understanding of the terms *codomain*, *domain* and *inverse function* would also be useful but is not essential. You will need to be able to solve a pair of *simultaneous equations*, manipulate arithmetic and

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algebraic *expressions* — including those involving *squares*, *square roots*, *brackets* and *ratios* — and to plot and interpret *graphs*. If you are uncertain about any of these terms, you should consult the *Glossary*, which will also indicate where in *FLAP* they are developed. The following *Ready to study questions* will help you to decide whether you need to review some of these topics before embarking on this module.

Question R1 Given that $x = a/h$, $y = b/h$, and $a^2 + b^2 = h^2$, rewrite the following expressions in terms of a, b and h, simplifying each as far as possible:

(a) $1/x$, (b) x/y, (c) $x^2 + y^2$.

Question R2 Solve the following simultaneous equations:

$$2x + 5y = 6 \quad \text{and} \quad 3x - 10y = 9$$

Question R3 What do we mean when we say that the position of an object is a *function* of time?

---

angle
degree
minute of arc
second of arc
right angle
perpendicular
normal
You will find the answers to these questions in Section 6.

2 Triangles and trigonometric ratios

2.1 Angular measure: degrees and radians

When two straight lines pass through a common point, the **angle** between the lines provides a measure of the inclination of one line with respect to the other or, equivalently, of the amount one line must be rotated about the common point in order to make it coincide with the other line. The two units commonly used to measure angles are *degrees* and *radians* (discussed below) and we will use both throughout this module. Greek letters, α (alpha), β (beta), γ (gamma), ... θ (theta), ϕ (phi) ... are often used to represent the values of angles, but this is not invariably the case.

A **degree** is defined as the unit of angular measure corresponding to 1/360th of a circle and is written as 1°. In other words, a rotation through 360° is a complete revolution, and an object rotated through 360° about a fixed point is returned to its original position. Fractions of an angle measured in degrees are often expressed as decimals, as in 97.8°, but it is also possible to use subsidiary units usually referred to as *minutes* and *seconds* for subdivisions of a degree, with the definitions that sixty minutes are equivalent to one degree and sixty seconds are equivalent to one minute. To distinguish them from units of time, these angular units are called the **minute of arc** and **second of arc**, abbreviated to arcmin and arcsec, respectively. The symbols $'$ and $''$ are often used for arcmin and arcsec, respectively. For example, $12'$ means 12 arcmin, and $35''$ means 35 arcsec.

An *arc* is a curve forming part of a circle — see later in this subsection for further details.

◆ (a) Express 6° 30′ as a decimal angle in degrees. (b) Express 7.2 arcmin in terms of arcsecs.

◇ (a) $30' = 0.5°$, so $6° 30' = 6.5°$.

(b) $7.2' = (7.2 \times 60)'' = 432''$. □

The angles 180° and 90° correspond to a rotation through half and one-quarter of a circle, respectively. An angle of 90° is known as a **right angle**. A line at 90° to a given line (or surface) is said to be **perpendicular** or **normal** to the original line (or surface).

It is conventional in mathematics and physics to refer to rotations in an anti-clockwise direction as *positive* rotations. So, a positive rotation through an angle θ would correspond to the anticlockwise movement shown in Figure 1, while a negative rotation of similar size would correspond to a clockwise movement. The negative rotation may be described as a rotation through an angle $-\theta$.

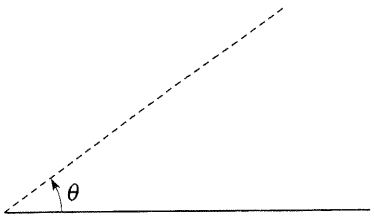


Figure 1 A positive rotation through a (positive) angle, θ.

An object which is rotated through an angle of 0° or 360° or 720° appears to remain unchanged and, in this sense, these rotations are equivalent. In the same sense rotations of 10°, 370°, 730°, −350° and so on are equivalent, since each can be obtained from the others by the addition of a multiple of 360°. When considering the orientational effect of a rotation through an angle θ it is only necessary to consider values of θ which lie in the range $0° \leq \theta < 360°$ since the orientational

The symbol < is read as 'is less than' and the symbol ≤ is read as 'is less than or equal to'.

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effect of every rotation is equivalent to a rotation lying in this range. For example, a rotation through $-1072°$ is equivalent to one through $-1072° + 3 \times 360° = 8°$. Notice that the range of non-equivalent rotations, $0° \leq \theta < 360°$, does *not* include 360°. This is because a rotation through 360° is equivalent to one through 0°, which *is* included.

◆ Find a rotation angle θ in the range $0° \leq \theta < 360°$ that has the equivalent orientational effect to each of the following: 423.6°, −3073.35° and 360°.

◇ $423.6° - 360° = 63.6°$; $-3073.35° + 9 \times 360° = 166.65°$; $360° - 360° = 0°$. □

Of course, angles that differ by a multiple of 360° are not equivalent in every way. For example, a wheel that rotates through 36 000° and thus completes 100 revolutions would have done something physically different from an identical wheel that only rotated through an angle of 360°, even if their final orientations were the same.

Despite the widespread use of degrees, a more natural (and important) unit of angular measure is the **radian**. As Figure 2 indicates, the radian may be defined as the angle subtended at the centre of a circle by an **arc** of the circle which has an **arc length** (☞) equal to the radius of the circle. As will be shown below, it follows from this definition that 1 radian (often abbreviated as 1 rad or sometimes 1^c) is equal to 57.30°, to two decimal places.

radian
arc
arc length
pi, π

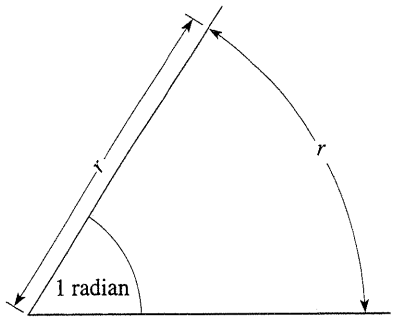


Figure 2 An angle of one radian.

An arc of a circle is a curve forming part of the circumference of that circle, and the arc length is the length of such a curve, measured along the curve. The meaning of 'subtended' can be seen from the figure.

Radians are such a natural and widely used unit of angular measure that whenever you see an angular quantity quoted without any indication of the associated unit of measurement, you should assume that the missing unit is the radian.

In general, as indicated in Figure 3, if an arc length s at a distance r subtends an angle ϕ at the centre of the circle, then the value of ϕ, measured in radians, is:

$$\phi = \frac{s}{r} \text{ rad} \qquad (1)$$

i.e. $\phi/\text{rad} = s/r$

Angular quantities such as ϕ generally represent the product of a number and a unit of angular measure such as a degree or a radian. Thus, the quotient ϕ/rad represents a pure number and may be read as 'the numerical value of ϕ measured in radians'.

This is a sensible definition of an angle since it is independent of the scale of Figure 3. For a given value of ϕ, a larger value of r would result in a larger value of s but the ratio s/r would be unchanged.

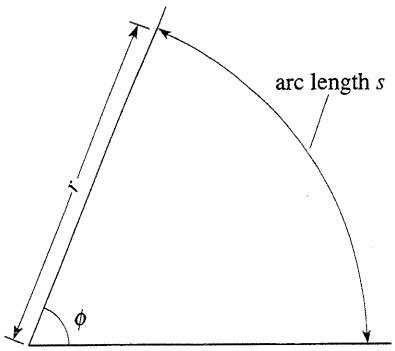


Figure 3 An angle measured in radians.

To determine the fixed relationship between radians and degrees it is probably easiest to consider the angle subtended at the centre of a circle by the complete circumference of that circle. A circle of radius r has a circumference of arc length $2\pi r$, where π represents the mathematical constant **pi**, an *irrational number* the value of which is 3.1416 to four decimal places. Consequently, the ratio of circumference to radius is 2π and the angle subtended at the centre of any circle by its circumference is 2π rad. But the complete angle at the centre of a circle is 360°, so we obtain the general relationship

2π radians = 360°

Note An *irrational number* is one that *cannot* be expressed as a quotient of two whole numbers, and hence extends to an infinite number of decimal places. See the *Glossary* for further details.

Since $2\pi = 6.2832$ (to four decimal places) it follows that 1 radian = 57.30°, as claimed earlier. Table 1 gives some angles measured in degrees and radians. As you can see from this table, many commonly-used angles are simple fractions or multiples of π radians, but note that angles expressed in radians are not *always* expressed in terms of π. Do not make the common mistake of thinking that π is some kind of angular unit; it is simply a number.

Table 1 Equivalent angles in degrees and radians.

Angle measured in degrees	Angle measured in radians
360	2π
180	π
90	$\pi/2$
60	$\pi/3$
45	$\pi/4$
30	$\pi/6$

◆ What is 10° expressed in radians?

◇ $\dfrac{10°}{180°} \times \pi \text{ rad} \approx \dfrac{\pi \text{ rad}}{18} \approx 0.175 \text{ rad}$. □

Question T1 The usual way of converting a quantity expressed in terms of one set of units into some other set of units is to multiply it by an appropriate *conversion factor*. For example, multiplying a distance measured in kilometres by the

You will find the answers to all *Text questions* in Section 6.

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conversion factor 1000 m km^{-1} will provide the equivalent distance in metres. What are the conversion factors from radians to degrees and from degrees to radians? □

Question T2 What is the non-negative rotation angle less than 2π rad that has the same orientational effect as a rotation through -201π rad? □

2.2 *Right-angled triangles*

Figure 4 shows a general triangle — a geometrical figure made up of three straight lines. There are three angles formed by the intersections of each pair of lines. By convention, the three angles are labelled according to the corresponding vertices as $\hat{A}, \hat{B}$ and $\hat{C}$. An alternative notation is to use $A\hat{B}C$ instead of $\hat{B}$, where AB and BC are the two lines defining the angle. In this module we use the $A\hat{B}C$ and $\hat{B}$ notations interchangeably. (☞) The angles $\hat{A}, \hat{B}$ and $\hat{C}$ in Figure 4 are known as the **interior angles** of the triangle.

interior angle
right-angled triangle
hypotenuse
Pythagoras's theorem

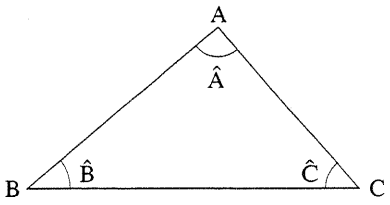


Figure 4 A general triangle.

$A\hat{B}C$ is spoken of as 'the angle ABC' and $\hat{B}$ as 'the angle B'.

Another alternative notation, not used in this module, for the angle $A\hat{B}C$ is ∠ABC.

The sum of the interior angles of *any* triangle is 180° (π radians). To see that this is so, imagine an arrow located at point A, superimposed on the line AB and pointing towards B. Rotate the arrow anticlockwise about the point A through an angle $\hat{A}$ so that it is aligned with AC and points towards C. Then rotate the arrow through an additional angle $\hat{C}$ so that it is horizontal and points to the right, parallel with BC. Finally, rotate the arrow through an additional angle $\hat{B}$ about point A. After this final rotation the arrow will again be aligned with the line AB, but will now point away from B having rotated through a total angle of 180°. So, rotating through $\hat{A} + \hat{B} + \hat{C}$ is equivalent to half a complete turn, i.e. 180°.

For the time being, we will be concerned only with the special class of triangles in which one interior angle is 90°. There are infinitely many such triangles, since the other two interior angles may have any values provided their sum is also 90°. Any such triangle is called a **right-angled triangle** (see Figure 5), and the side opposite the right angle is known as the **hypotenuse**. Notice the use of the box symbol, ⌞, to denote a right angle. Right-angled triangles are important because:

- There is a special relationship between the lengths of the sides (Pythagoras's theorem).
- There is a strong link between right-angled triangles and the trigonometric ratios (sine, cosine and tangent).
- Applications involving right-angled triangles often occur in physics (and many other branches of science and engineering).

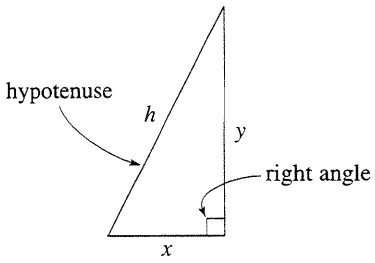


Figure 5 A right-angled triangle with sides of length x, y and h.

We shall consider each of these points in this module, but the rest of this subsection will be devoted to Pythagoras's theorem.

> **Pythagoras's theorem** states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.
>
> i.e. $h^2 = x^2 + y^2$

Consideration of Figure 6 shows how this result comes about for a general right-angled triangle of sides x, y and h. One way of finding the area of the large outer square is by squaring the length of its sides, i.e.

$$\text{area} = (x+y)^2 = x^2 + y^2 + 2xy \qquad (2)$$

However, the area of the large square can also be found by adding the area of the smaller square, h^2, to the areas of the four corner triangles. Each triangle has an area $xy/2$ (each is half a rectangle of sides x and y) so the area of the large square is

$$\text{area} = h^2 + 4xy/2 = h^2 + 2xy \qquad (3)$$

Comparing the right-hand sides of Equations 2 and 3 shows that

$$h^2 = x^2 + y^2 \qquad (4)$$

which is Pythagoras's theorem.

The *converse* of Pythagoras's theorem is also true; that is, if the sum of the squares of two sides of a triangle is equal to the square of the other side, then the triangle is

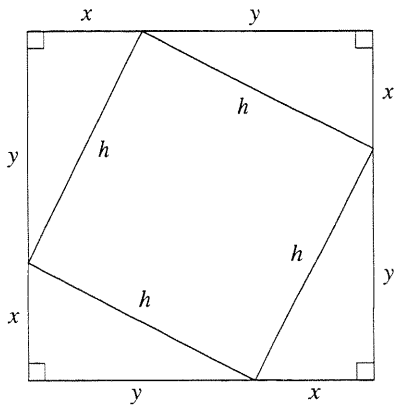


Figure 6 Pythagoras's theorem can be derived by writing expressions for the area of the large square.

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right-angled. For the purpose of this module, we will accept the validity of the converse without proving it.

similar triangles
opposite side
adjacent side

◆ The angles of a triangle are 45°, 90° and 45° and two of the sides (that is, those sides opposite the 45° angles) have lengths of 10 m. What is the length of the hypotenuse?

◇ By Pythagoras's theorem, the length of the hypotenuse is $\sqrt{10^2 + 10^2}$ m = $\sqrt{200}$ m = 14.14 m (to two decimal places). □

◆ The hypotenuse of a right-angled triangle is 7 m long while one of the other sides is of length 3 m. What is the length of the remaining side?

◇ If we denote the length of the remaining side by x, then, from Pythagoras's theorem, $7^2\ m^2 = 3^2\ m^2 + x^2$, so that $x^2 = (7^2 - 3^2)\ m^2 = (49 - 9)\ m^2 = 40\ m^2$ and therefore $x = \sqrt{40}$ m = 6.32 m (to two decimal places). □

The sides of some right-angled triangles can be expressed entirely in terms of integers; probably the most famous is the 3 : 4 : 5 triangle where the hypotenuse has length 5 units and the other two sides have lengths 3 and 4 units: $3^2 + 4^2 = 5^2$.

Note Throughout the remainder of this module we will not usually express lengths in any particular units. This is because we are generally interested only in the *ratios* of lengths. Of course, when you are considering real physical situations, you must attach appropriate units to lengths.

Question T3 Show that triangles with sides in the ratios 5 : 12 : 13 and 8 : 15 : 17 are also right-angled triangles. □

Question T4 Use Pythagoras's theorem to show that the hypotenuse is always the longest side of a right-angled triangle. (*Hint*: Consider $a^2 = b^2 + c^2$.) □

2.3 The trigonometric ratios

In the previous subsection, we indicated that the ratios of the lengths of the sides of a triangle were often of more interest than the actual lengths themselves. Figure 7 shows some **similar triangles**, i.e. triangles that are the same shape but different sizes — in other words, triangles with corresponding angles that are equal but with corresponding sides of different lengths. Although the *lengths* of the sides of any one triangle may differ from those of any similar triangle, the *ratios* of the side lengths are the same in each triangle — for example, each triangle in Figure 7 has sides whose lengths are in the ratio 2 : 3 : 4. In future, whenever we say that two or more triangles are similar we will mean it in the technical sense that they have the same interior angles and side lengths that are in the same ratio.

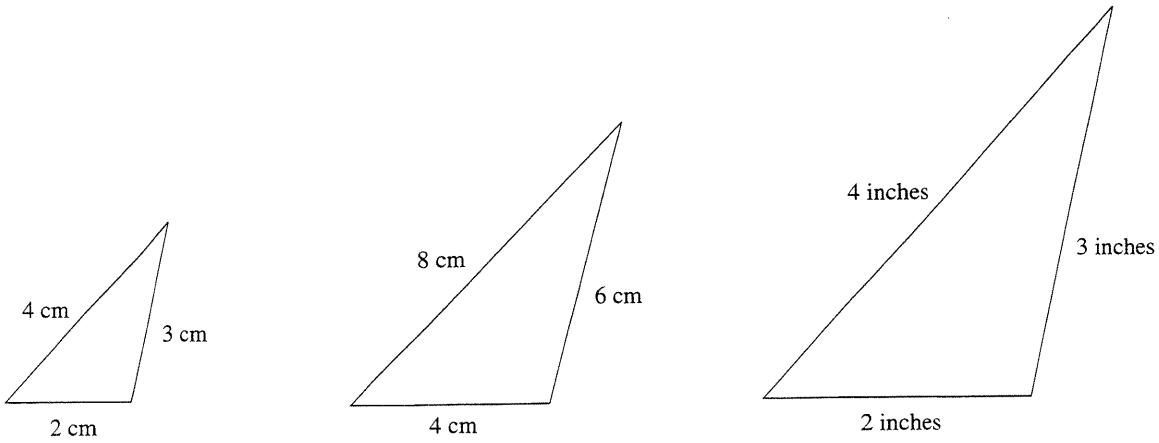


Figure 7 Some similar triangles. Each triangle has the same interior angles, and the lengths of the sides of any one triangle are in the same ratio to one another (2 : 3 : 4 in this case) as the lengths of the sides of any of the other similar triangles. (These are not to scale.)

Figure 8 shows a right-angled triangle in which an angle θ has been marked for particular attention and the **opposite side** and **adjacent side** to this angle have been

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identified. Thus the three sides may be referred to as the opposite, the adjacent and the hypotenuse, and we may use these terms or the letters o, a and h to refer to their respective lengths. Thanks to the special properties of right-angled triangles, the whole class of triangles that are similar to the triangle in Figure 8 can be characterized by the single angle θ, or, equivalently, by the ratio of the side lengths $o : a : h$. In fact, because of the Pythagorean relationship that exists between the sides of a right-angled triangle, the ratio of any *two* side lengths is sufficient to determine θ and identify the class of similar triangles to which a given right-angled triangle belongs. The ratios of the sides of right-angled triangles are therefore of particular importance. The study of right-angled triangles is known as **trigonometry**, and the three distinct ratios of pairs of sides are collectively known as the **trigonometric ratios**. They are called the **sine**, **cosine** and **tangent** of the angle θ—abbreviated to sin, cos and tan, respectively —and defined as follows:

trigonometry
trigonometric ratios
sine
cosine
tangent

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{6}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \tag{7}$$

The abbreviations used for trigonometric ratios are not entirely standard. Some texts, for example, abbreviate tangent as tg.

Given three side lengths o, a and h, it is possible to form *six* different ratios; o/h, a/h, o/a, h/o, h/a and a/o. However, the last three of these are merely the reciprocals of the first three and are therefore known as the *reciprocal trigonometric ratios*. Nonetheless, they are of some interest and are dealt with in the next subsection.

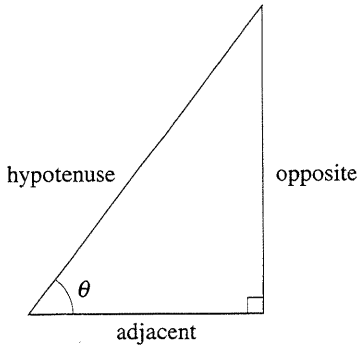


Figure 8 The labelling of sides in a right-angled triangle.

The angle θ that appears in these definitions must lie between 0 rad (0°) and $\pi/2$ rad (90°), but later in this module we will extend the definitions to all angles. It should be emphasized that the value of a particular trigonometric ratio depends only on the value of θ, so the sine, cosine and tangent are effectively *functions* of the angle θ. More will be made of this point in Section 3 when we actually define related quantities called the *trigonometric functions*.

It is extremely useful to remember the definitions of the trigonometric ratios. You may find it helpful to denote the sine, cosine and tangent by the letters s, c and t and then, using h, o and a to represent hypotenuse, opposite and adjacent, the three relations read, left to right and top to bottom, as *soh*, *cah* and *toa*.

You can use a calculator to find the sine, cosine or tangent of an angle expressed in either degrees or radians, provided you first switch it to the appropriate mode —this is usually done by pressing a key marked 'DRG' (or something similar) until either 'degrees' or 'radians' appears in the display. Then key in the angle followed by one of the function keys sin, cos or tan.

◆ Use a calculator to find sin (15°); cos (72°); sin ($\pi/4$); tan ($\pi/3$); cos (0.63).

◇ sin (15°) = 0.2588; cos (72°) = 0.3090; sin ($\pi/4$) = 0.7071; tan ($\pi/3$) = 1.732; cos (0.63) = 0.8080. (Note that the last three angles should be assumed to be in radians since no units are stated.) □

The three trigonometric ratios are not all independent since we can write:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} = \sin\theta \times \frac{1}{\cos\theta}$$

and so we obtain the identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{8}$$

Furthermore, we can use Pythagoras's theorem to obtain a second relation. Starting with:

$$(\text{opposite})^2 + (\text{adjacent})^2 = (\text{hypotenuse})^2$$

we can divide both sides by $(\text{hypotenuse})^2$ to obtain

$$\frac{(\text{opposite})^2}{(\text{hypotenuse})^2} + \frac{(\text{adjacent})^2}{(\text{hypotenuse})^2} = 1$$

and therefore

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Note that we have used brackets to distinguish, for example, $\tan\frac{\pi}{3}$ from $\frac{\tan\pi}{3}$. Never hesitate to use brackets in this way if it will reduce the chance of confusion.

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As you can see, writing powers of trigonometric functions can be rather cumbersome and so the convention that $\sin^n \theta$ means $(\sin \theta)^n$ (for *positive* values of n) is often used. Similar conventions are used for the other trigonometric functions. The notation cannot be used for negative values of n since $\sin^{-1} \theta$ is sometimes used for the inverse sine function, which we consider later in this module. The above relation can therefore be written as:

$$\sin^2 \theta + \cos^2 \theta = 1 \qquad (9)$$

Since there are two identities relating the trigonometric ratios, it follows that only one ratio is independent and therefore given one ratio we can find the other two. (This assumes that the trigonometric ratios are positive, which is true for $0° \leq \theta \leq 90°$.)

The angle ϕ in Figure 9 also has its sine, cosine and tangent. But the opposite and adjacent sides appropriate to θ are interchanged for ϕ and, as a consequence we can write

$$\tan \theta = \frac{1}{\tan \phi}, \sin \theta = \cos \phi, \cos \theta = \sin \phi$$

There are some special angles for which it is easy to write down the sine, cosine and tangent. As an example, consider the right-angled triangle with two sides of equal length, as shown in Figure 10.

Any triangle with two sides of equal length is called an **isosceles triangle**, and any isosceles triangle must contain two equal interior angles. The isosceles triangle of Figure 10 is special because it is also a right-angled triangle. Since the interior angles of any triangle add up to 180°, the angles of this particular triangle must be 45°, 90°, 45°. Also, since the two equal sides of this particular triangle are both of unit length it follows from Pythagoras's theorem that the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$ and so we can write down the following results:

$$\sin(45°) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\cos(45°) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

$$\tan(45°) = \frac{\text{opposite}}{\text{adjacent}} = 1$$

Figure 11 shows an **equilateral triangle**, i.e. one with three sides of equal length and hence three equal interior angles which must be equal to 60°. A line has been drawn from one vertex (i.e. corner) to the middle of the opposite side, so that the angle between the line and the side is 90° (that is, the line is a *normal* to the side).

Question T5 By considering Figure 11, find the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ for θ equal to 30° ($\pi/6$ rad) and 60° ($\pi/3$ rad), and hence complete the trigonometric ratios in Table 2. □

Table 2 See Question T5.

θ /degrees	θ /radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	0	1	0
30	$\pi/6$			
45	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60	$\pi/3$			
90	$\pi/2$	1	0	undefined

Question T6 Figure 12 shows a graph of $\sin \theta$ for $0 \leq \theta < \pi/2$. Using Table 2, your answer to Question T5, and any other relevant information given in this subsection, sketch corresponding graphs for $\cos \theta$ and $\tan \theta$. □

One of the many reasons why trigonometric ratios are of interest to physicists is that they make it possible to determine the lengths of all the sides of a right-angled triangle from a knowledge of just one side length and one interior angle (other than the right angle). Here is an example of this procedure taken from the field of optics.

isoceles triangle
equilateral triangle

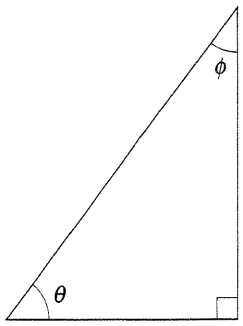


Figure 9 A right-angled triangle with angles θ and ϕ.

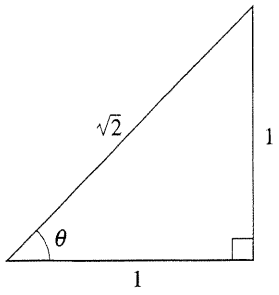


Figure 10 A right-angled triangle with two equal length sides, an isosceles triangle.

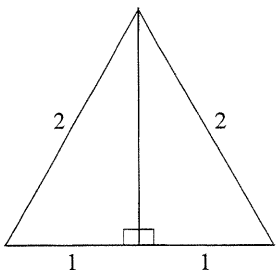


Figure 11 An equilateral triangle.

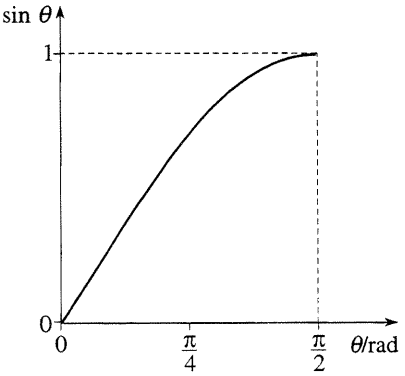


Figure 12 See Question T6.

In order to sketch the graphs in Question T6, we are assuming that the curve is smooth between the points where we have calculated its value.

The trigonometric functions can be evaluated directly using expressions involving θ. For details, see the *FLAP* module that deals with *expansions* and *approximations*.

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Question T7 Figure 13 represents a two-slit diffraction experiment. Derive a formula relating λ to d and θ. If $d = 2 \times 10^{-6}$ m and $\theta = 17.46°$, what is the value of λ? □

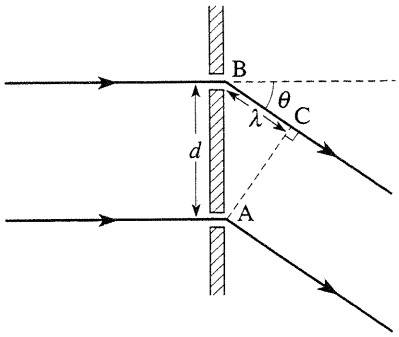


Figure 13 See Question T7.

reciprocal trigonometric ratios
cosecant
secant
cotangent

Diffraction is discussed in the physics strand of *FLAP*. You can find out more about it from the *Glossary*, but you do not need to know the details to answer this question.

2.4 The reciprocal trigonometric ratios

The ratios introduced in the previous subsection could all have been written the other way up. The resulting **reciprocal trigonometric ratios** occur so frequently that they too are given specific names; they are the **cosecant**, **secant**, and **cotangent** (abbreviated to cosec, sec and cot) and are defined by:

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \quad \text{provided } \sin\theta \neq 0 \tag{10}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{provided } \cos\theta \neq 0 \tag{11}$$

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{provided } \tan\theta \neq 0 \tag{12}$$

An alternative abbreviation for cosec is csc.

Notice that cosec is the reciprocal of sin, and sec the reciprocal of cos. This terminology may seem rather odd but it is easily remembered by recalling that each reciprocal pair — (sin, cosec), (cos, sec), (tan, cot) — involves the letters 'co' just once. In other words there is just one 'co' between each pair. Also notice that each reciprocal trigonometric function is undefined when its partner function is zero.

Throughout the domains on which they are defined, each of the reciprocal trigonometric ratios can also be written in terms of the sides of the triangle in Figure 8:

$$\text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{opposite}} \tag{13}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \tag{14}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} \tag{15}$$

Two useful general relationships follow from Equations 13–15 and Pythagoras's theorem:

$$\cot\theta = \frac{\text{cosec}\,\theta}{\sec\theta} \tag{16}$$

and $$1 + \tan^2\theta = \sec^2\theta \tag{17}$$

Question T8 Using Figure 11, write down the values of cosec (30°), sec (30°), cot (30°) and cosec (60°), sec (60°), cot (60°). □

Calculators do not generally have keys that give the reciprocal trigonometric ratios directly, but the ratios can be found using the sin, cos and tan keys and the reciprocal ($1/x$) key.

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Question T9 Use a calculator to find cosec (23°), sec (56°), cot (π/6), cot (1.5). □

Question T10 Figure 14 shows a graph of cosec θ for $0 < \theta < \pi/2$. Using values of reciprocal trigonometric ratios calculated above, and other information from this subsection, sketch graphs of sec θ and cot θ for $0 \leq \theta < \pi/2$. □

trigonometric functions

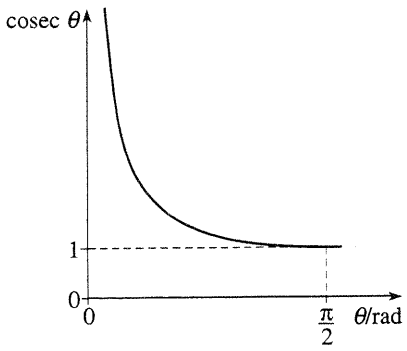


Figure 14 See Question T10.

2.5 Small angle approximations

We end this section with some useful approximations involving small angles. Figure 15 shows a right-angled triangle with one very small angle θ and the third angle almost a right angle. If θ is at the centre of a circle radius r, where r is the hypotenuse of the triangle, you can see from the diagram that the opposite side to θ is almost coincident with the arc length s and the adjacent side to θ is almost the same length as the hypotenuse. From Equation 1, s/r is the value of θ *in radians*. So, for the small angle θ, Equations 5 to 7 give $\sin\theta \approx s/r$, $\cos\theta \approx r/r$, $\tan\theta \approx s/r$ and hence:

for a small angle θ	$\cos\theta \approx 1$
and	$\sin\theta \approx \tan\theta \approx \theta/\text{rad}$

◆ Use a calculator to find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for a few small angles, and hence show that the approximations expressed in the boxed equations above become increasingly good as θ becomes smaller. Try, for example, $\theta = 0.175\,00$ rad (i.e. $\theta \approx 10°$) and $\theta = 0.010\,00$ rad, and express the answers to five decimal places.

◇ $\sin(0.175\,00) = 0.174\,11$; $\cos(0.175\,00) = 0.984\,73$; $\tan(0.175\,00) = 0.176\,81$. $\sin(0.010\,00) = 0.010\,00$; $\cos(0.010\,00) = 0.999\,95$; $\tan(0.010\,00) = 0.010\,00$. □

◆ Write down approximate expressions for the reciprocal trigonometric ratios for a small angle θ.

◇ $\text{cosec}\,\theta \approx \cot\theta \approx 1/(\theta/\text{rad})$; $\sec\theta \approx 1$. □

Question T11 Seen from Earth, the diameter of the Sun subtends an angle ϕ of about 0.5°. By expressing ϕ in radians, derive an expression for the Sun's diameter, s, in terms of its distance d from Earth. Your expression should *not* involve any trigonometric ratios. □

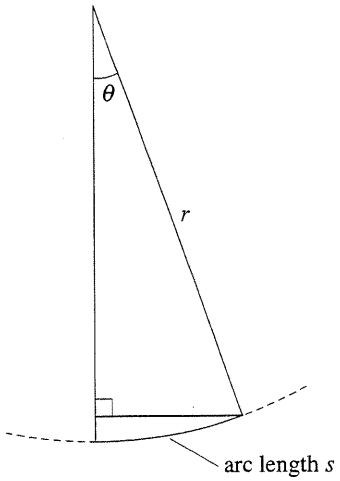


Figure 15 A right-angled triangle with a small angle θ.

3 Graphs and trigonometric functions

3.1 The trigonometric functions

The ratio definitions of the sine, cosine and tangent (i.e. Equations 5, 6 and 7) only make sense for angles in the range 0 to $\pi/2$ radians, since they involve the sides of a right-angled triangle. In this subsection we will define three **trigonometric functions**, also called sine, cosine and tangent, and denoted $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$, respectively. These functions will enable us to attach a meaning to the sine and cosine of *any* angle, and to the tangent of any angle that is not an odd multiple $\pi/2$. Like the trigonometric ratios that they generalize, these trigonometric functions are of great importance in physics.

Throughout this section we will be careful to use brackets (as in $\sin(\theta)$) to distinguish the trigonometric functions from the trigonometric ratios ($\sin\theta$, etc.), but, since the trigonometric functions and ratios agree in those regions where they are both defined, this distinction is of little importance in practice.

In defining the trigonometric functions we want to ensure that they will agree with the trigonometric ratios over the range 0 to $\pi/2$ radians. In order to do this, consider the point P shown in Figure 16 that moves on a circular path of unit radius around the origin O of a set of two-dimensional Cartesian coordinates (x, y). If we let θ be the angle between the line OP and the x-axis, it follows from the definition of the trigonometric ratios that the coordinates of P are

$$y = \sin\theta \quad \text{for } 0 \leq \theta < \pi/2$$

and $$x = \cos\theta \quad \text{for } 0 \leq \theta < \pi/2$$

Although these trigonometric ratios are only defined over a very narrow range, it is easy to imagine the angle θ increasing in the positive (i.e. anticlockwise) direction

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to take up *any* positive value, with P crossing the positive x-axis whenever θ is equal to an integer multiple of 2π. And it is equally easy to imagine θ increasing in the negative (clockwise) direction to take up *any* negative value. Now, whatever the value of θ may be, large or small, positive or negative, the point P must still be located somewhere on the circle in Figure 16 and it must have a single x- and a single y-coordinate. We can use the particular values of x and y that correspond to a given value of θ to *define* the first two trigonometric functions

$$\sin(\theta) = y \quad \text{for any } \theta$$

and $$\cos(\theta) = x \quad \text{for any } \theta$$

Defined in this way, it is inevitable that the trigonometric functions will agree with the trigonometric ratios when $0 \leq \theta \leq \pi/2$, but it is also clear that the functions, unlike the ratios, make sense for arbitrary values of θ.

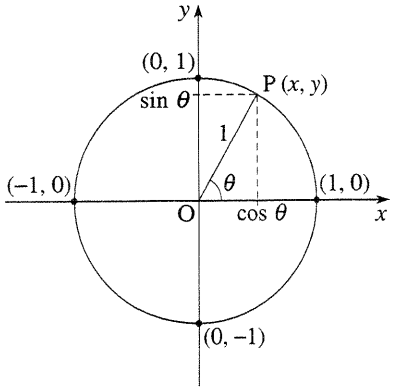


Figure 16 Defining the trigonometric functions for any angle. If $0 \leq \theta < \pi/2$, the coordinates of P are $x = \cos\theta$ and $y = \sin\theta$. For general values of θ we define $\sin(\theta) = y$ and $\cos(\theta) = x$.

Having defined the functions $\sin(\theta)$ and $\cos(\theta)$ we also want to define a function $\tan(\theta)$, but once again we want to ensure consistency with the behaviour of the trigonometric ratio $\tan\theta$ that was introduced earlier. We can do this by using a generalization of Equation 8 (i.e. $\tan\theta = \sin\theta/\cos\theta$) as the basis of the definition. Thus, we define

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \text{for any } \theta \neq (2n+1)\pi/2$$

If n may be any positive or negative whole number, $(2n + 1)$ represents an odd integer. So the condition on the right of the definition is a way of saying that the equation applies for any value of θ that is *not* an odd multiple of $\pi/2$.

Notice that unlike the sine and cosine functions, this function is not defined for values of θ that are odd integer multiples of $\pi/2$. This restriction is imposed because $\cos\theta = 0$ at those values of θ, and the quotient $\sin(\theta)/\cos(\theta)$ has no meaning. (☞)

1/0 has no meaning. It is not even correct to say that it is infinity.

Having defined the trigonometric functions it is only natural to enquire about their graphs, since graphs are usually a good way of gaining insight into the behaviour of functions. Perhaps the first thing to notice about the trigonometric functions is that they are not always positive. If θ is in the range $0 \leq \theta < \pi/2$ both the x- and y-coordinates of P will be positive, so both $\sin(\theta)$ and $\cos(\theta)$ will be positive as will their quotient $\tan(\theta)$. However, as θ enters the range $\pi/2 < \theta < \pi$ the x-coordinate of P becomes negative, so $\cos(\theta)$ and $\tan(\theta)$ will be negative, though $\sin(\theta)$ will remain positive. Similarly, when $\pi < \theta < 3\pi/2$, x and y are both negative so $\sin(\theta)$ and $\cos(\theta)$ are negative while $\tan(\theta)$ is positive; and if $3\pi/2 < \theta < 2\pi$, $\cos(\theta)$ is positive while $\sin(\theta)$ and $\tan(\theta)$ are negative. As θ increases beyond 2π (or when θ decreases below 0) the same pattern is repeated.

A *quadrant* is a quarter of a circle.

Figure 17 summarizes the sign behaviour of the trigonometric functions. The positive function in each quadrant is indicated by its initial letter, or in the case of the first quadrant where all the functions are positive by the letter A. Most people who use the trigonometric functions find it helpful to memorize Figure 17 (or to remember Figure 16, so that they can work it out). A traditional mnemonic to help recall which letter goes in which quadrant is 'All Stations To Crewe', which gives the letters in positive (anticlockwise) order starting from the first quadrant.

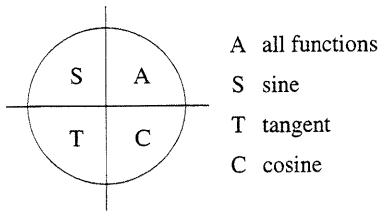


Figure 17 Quadrants in which the trigonometric functions are positive.

Of course, it is not only the signs of the trigonometric functions that change as θ increases or decreases and P moves around the circle in Figure 16. The *values* of x and y, and consequently of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ also vary.

◆ If you were asked to draw a diagram similar to Figure 17, but showing which trigonometric function(s) increase as θ increases in each quadrant, how would you have to change the lettering on Figure 17.

◇ A would become S, T (both $\sin(\theta)$ and $\tan(\theta)$ are increasing from zero in the first quadrant.)

S would become T (as $\sin(\theta)$ decreases you might think that $\tan(\theta)$ would also decrease, but $\cos(\theta)$ is negative and decreasing in the second quadrant so $\tan(\theta)$ becomes a smaller negative number as θ increases, i.e. the value of $\tan(\theta)$ *increases*.)

T would become C, T (in this quadrant it is $\cos(\theta)$ that is becoming less negative.)

C would become A ($\sin(\theta)$ and $\tan(\theta)$ are both becoming less negative and $\cos(\theta)$ is increasing from zero in this quadrant.) □

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The graphs of sin (θ), cos (θ) and tan (θ) are shown in Figures 18, 19 and 20. As you can see, the values sin (θ) and cos (θ) are always in the range -1 to 1, and any given value is repeated each time θ increases or decreases by 2π. The graph of tan (θ) is quite different. Values of tan (θ) cover the full range of real numbers, but tan (θ) tends towards $+\infty$ as θ approaches odd multiples of $\pi/2$ from below, and towards $-\infty$ as θ approaches odd multiples of $\pi/2$ from above. This emphasizes the impossibility of assigning a meaningful value to tan (θ) at odd multiples of $\pi/2$. Table 3 lists the values of the trigonometric functions for some angles between 0 and 2π.

The symbol ∞ means 'infinity'.

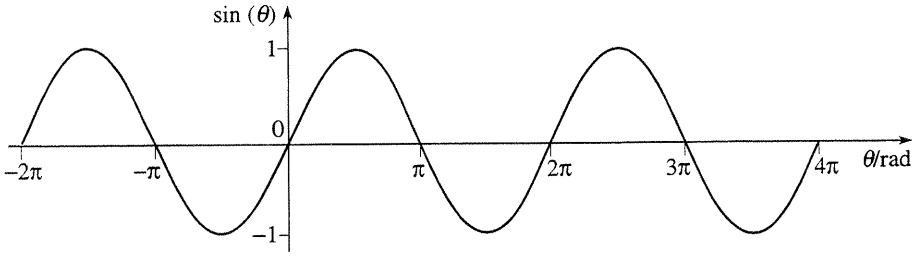


Figure 18 Graph of sin (θ).

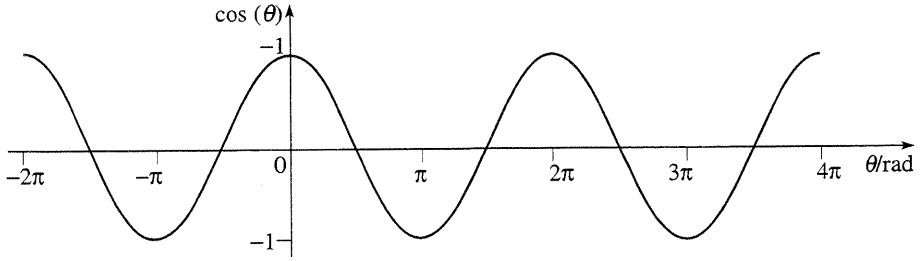


Figure 19 Graph of cos (θ).

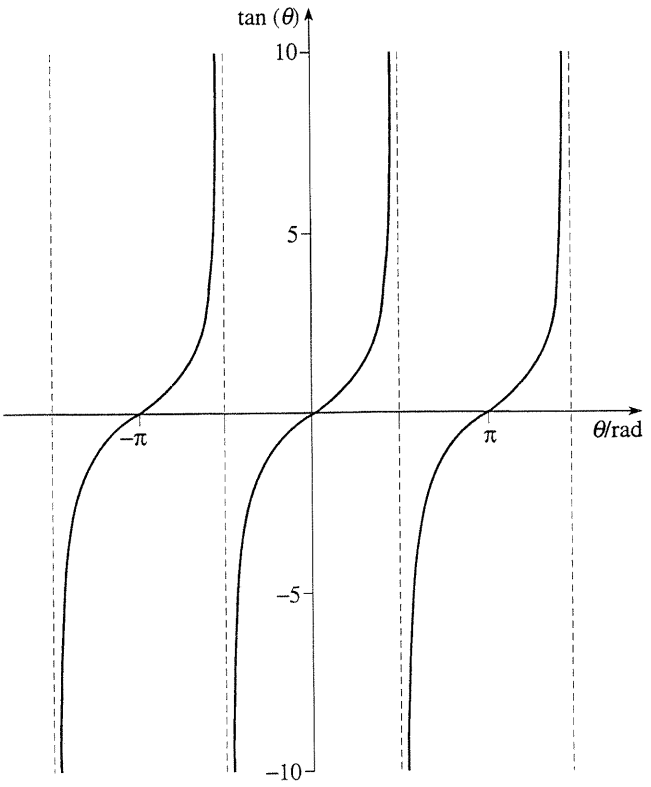


Figure 20 Graph of tan (θ).

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reciprocal trigonometric functions

Table 3 The trigonometric functions between 0 and 2π.

θ/radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
$\pi/2$	1	0	undefined
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$3\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1
$5\pi/6$	1/2	$-\sqrt{3}/2$	$-1/\sqrt{3}$
π	0	-1	0
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3}$
$5\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
$3\pi/2$	-1	0	undefined
$5\pi/3$	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$
$7\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1
$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$
2π	0	1	0

Question T12 Describe as many significant features as you can of the graphs in Figures 18 and 19. (Some of these features will be discussed in the next subsection.) □

Given the trigonometric functions, we can also define three **reciprocal trigonometric functions** cosec (θ), sec (θ) and cot (θ), that generalize the reciprocal trigonometric ratios defined in Equations 10, 11 and 12. The definitions are straightforward, but a little care is needed in identifying the appropriate domain of definition in each case. (As usual we must choose the domain in such a way that we are not required to divide by zero at any value of θ.)

$$\text{cosec}(\theta) = \frac{1}{\sin(\theta)} \qquad \theta \neq n\pi$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \theta \neq (2n+1)\pi/2$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \qquad \theta \neq n\pi$$

Graphs of the reciprocal trigonometric functions are shown in Figures 21, 22 and 23.

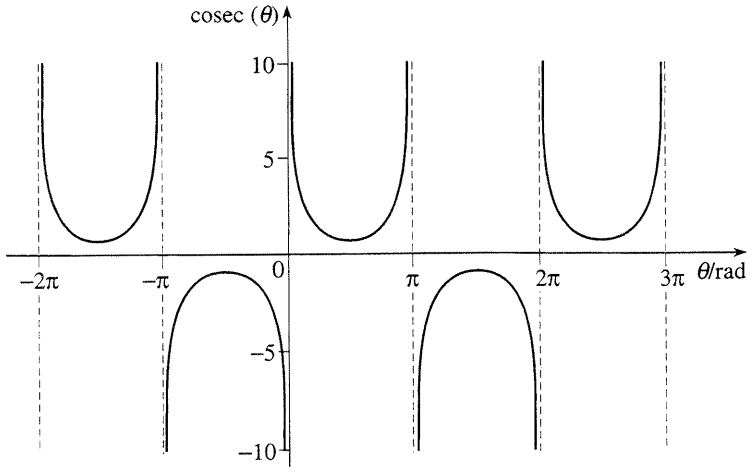


Figure 21 Graph of cosec (θ).

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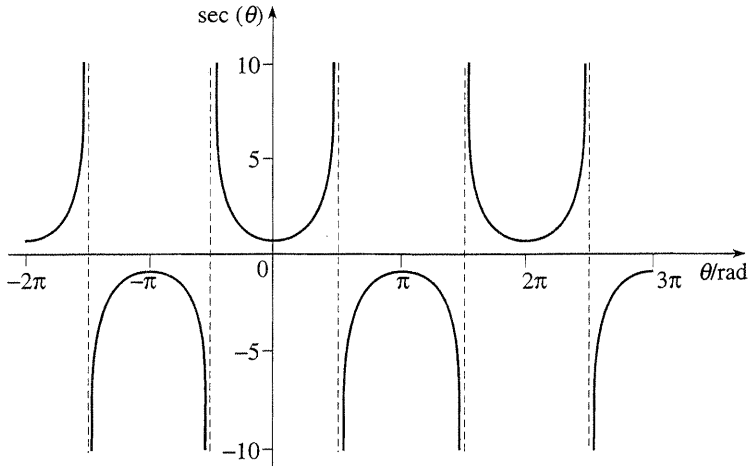


Figure 22 Graph of sec (θ).

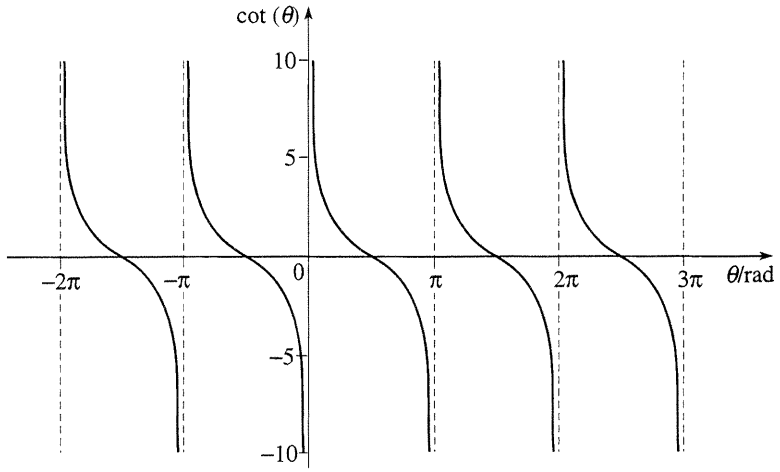


Figure 23 Graph of cot (θ).

Throughout this subsection the argument θ of the various trigonometric and reciprocal trigonometric functions has always been an angle measured in radians. (This is true even though we have been conventionally careless about making sure that we always include the appropriate angular unit when assigning numerical values to θ.) However, the arguments of these functions do not *have* to be angles. If we regarded the numbers printed along the horizontal axes of Figures 18 to 23 as values of a purely numerical variable, x say, rather than values of θ in radians, we could regard the graphs as *defining* six functions of x; $\sin(x)$, $\cos(x)$, $\tan(x)$, etc. Strictly speaking these new functions are quite different from the trigonometric functions (☞) and should be given different names to avoid confusion. But, given the tendency of physicists to be careless about domains and their habit of 'dropping' the explicit mention of radian from angular values, there is no practical difference between these new functions and the true trigonometric functions, so the confusion of names is harmless. Nonetheless, it is worth remembering that what appears as the argument of a trigonometric function is not necessarily an angle.

The functions are different because their domains are different; a set of angles in the case of the trigonometric functions, and a set of real numbers in the case of the new functions sin (x), etc.

A common example of this arises in the study of oscillations where trigonometric functions are used to describe repeated back and forth motion along a straight line. In the simplest such motion, *simple harmonic motion*, the changing position x of a mass oscillating on the end of a spring may be represented by $x = A\cos(\omega t + \phi)$. Despite appearances none of the quantities inside the bracket is an angle (though they may be given angular interpretations); t is the time and is measured in seconds, ω is a constant known as the *angular frequency* that is related to the properties of the mass and spring and is measured in hertz (1 Hz = 1 s^{-1}), and ϕ, the *phase constant*, is a number, usually in the range 0 to 2π.

Oscillations are discussed in the physics strand of *FLAP*. See the *Glossary* for details.

One final point to note. As mentioned before, throughout this subsection we have been careful to use brackets (as in sin (θ)) to distinguish the trigonometric functions from the trigonometric ratios (sin θ, etc.), but since the trigonometric functions and

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ratios agree in those regions where they are both defined this distinction is also of little importance in practice. Consequently, as a matter of convenience, the brackets are usually omitted from the trigonometric functions unless such an omission is likely to cause confusion. In much of what follows we too will omit them and simply write the trigonometric and reciprocal trigonometric functions as sin x, cos x, tan x, cosec x, sec x and cot x.

periodic function
periodicity
period
even function
symmetric function
odd function
antisymmetric function

3.2 Periodicity and symmetry

The trigonometric functions are all examples of **periodic functions**. That is, as θ increases steadily, the same sets of values are 'recycled' many times over, always repeating exactly the same pattern. The graphs in Figures 18, 19 and 20, show this repetition, known as **periodicity**, clearly. More formally, a periodic function $f(x)$ is one which satisfies the condition $f(x) = f(x + nk)$ for every integer n, where k is a constant, known as the **period**.

Note that when we are dealing with a function of a sum of two angles the angle *must* be written in brackets to avoid ambiguity.

Adding or subtracting any multiple of 2π to an angle is equivalent to performing any number of complete rotations in Figure 16, and so does not change the value of the sine or cosine:

$$\sin(\theta) = \sin(\theta + 2n\pi) \quad \text{and} \quad \cos(\theta) = \cos(\theta + 2n\pi)$$

for any integer n. The functions $\sin(\theta)$ and $\cos(\theta)$, are therefore *periodic functions* with *period* 2π.

◆ What is the period of $\tan(\theta)$?

◇ Since $\tan(\theta) = \sin(\theta)/\cos(\theta)$ (if $\cos(\theta)$ is non–zero) it is tempting to say that $\tan(\theta)$ has period 2π, but we can actually do better than this. From Figure 20, and you can see that it repeats with a period of π. You can see why this is so if you look at Figure 16. Rotating P through π radians leaves the sizes of x and y unchanged, but changes the sign of both of them, with the result that $\tan\theta (= y/x)$ will be unaffected. □

As noted in the answer to Question T12, the trigonometric functions have some symmetry either side of $\theta = 0$. From Figures 18, 19 and 20 we can see the effect of changing the sign of θ:

$$\sin(-\theta) = -\sin(\theta) \tag{18}$$

$$\cos(-\theta) = \cos(\theta) \tag{19}$$

$$\tan(-\theta) = -\tan(\theta) \tag{20}$$

Any function $f(x)$ for which $f(-x) = f(x)$ is said to be **even** or **symmetric**, and will have a graph that is symmetrical about $x = 0$. Any function for which $f(-x) = -f(x)$ is said to be **odd** or **antisymmetric**, and will have a graph in which the portion of the curve in the region $x < 0$ appears to have been obtained by reflecting the curve for $x > 0$ in the vertical axis and then reflecting the resulting curve in the horizontal axis. It follows from Equations 18, 19 and 20 that $\cos(\theta)$ is an even function, while $\sin(\theta)$ and $\tan(\theta)$ are both odd functions.

An arbitrary function $f(x)$ is not necessarily even or odd, but any function can always be written as a sum of even and odd functions (see *odd* and *even functions* in the *Glossary*).

◆ For each of the reciprocal trigonometric functions, state the period and determine whether the function is odd or even.

◇ $\operatorname{cosec}(\theta)$ is an odd function with period 2π.

$\sec(\theta)$ is an even function with period 2π.

$\cot(\theta)$ is an odd function with period π. □

The combination of periodicity with symmetry or antisymmetry leads to further relationships between the trigonometric functions. For example, from Figures 18 and 19 you should be able to see that the following relationships hold true:

$$\sin(\theta) = \sin(\pi - \theta) = -\sin(\pi + \theta) = -\sin(2\pi - \theta) \tag{21}$$

and

$$\cos(\theta) = -\cos(\pi - \theta) = -\cos(\pi + \theta) = \cos(2\pi - \theta) \tag{22}$$

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phase constant
sinusoidal
inverse trigonometric functions

Less apparent, but just as true are the following relationships

$$\cos(\theta) = \sin(\pi/2 - \theta) = \sin(\pi/2 + \theta) = -\sin(3\pi/2 - \theta) = -\sin(3\pi/2 + \theta) \quad (23)$$

$$\sin(\theta) = \cos(\pi/2 - \theta) = -\cos(\pi/2 + \theta) = -\cos(3\pi/2 - \theta) = \cos(3\pi/2 + \theta) \quad (24)$$

Thanks to periodicity, all of these relationships (Equations 21 to 24) remain true if we replace any of the occurrences of θ by $(\theta + 2n\pi)$, where n is any integer.

It is quite clear from Figures 18 and 19 that there must be a simple relationship between the functions sin θ and cos θ; the graphs have exactly the same shape, one is just shifted horizontally relative to the other through a distance $\pi/2$. Equations 23 and 24 provide several equivalent ways of describing this relationship algebraically, but perhaps the simplest is that given by the first and third terms of Equation 23:

i.e. $\sin(\theta + \pi/2) = \cos(\theta)$

Clearly, adding a positive constant, $\pi/2$, to the argument of the function has the effect of shifting the graph to the left by $\pi/2$. In crude terms, the addition has boosted the argument and makes everything happen earlier (i.e. further to the left). Simple as it is, this is just one example of an important general principle that has many physical applications and deserves special emphasis.

> Adding *any* positive constant ϕ to θ has the effect of shifting the graphs of sin θ and cos θ horizontally to the left by ϕ, leaving their overall shape unchanged. Similarly, subtracting ϕ shifts the graphs to the right. The constant ϕ is known as the **phase constant**.

Since the addition of a phase constant shifts a graph but does not change its shape, all graphs of sin $(\theta + \phi)$ and cos $(\theta + \phi)$ have the same 'wavy' shape, regardless of the value of ϕ: any function that gives a curve of this shape, or the curve itself, is said to be **sinusoidal**.

◆ Sketch the graphs of sin (θ) and sin $(\theta + \phi)$ where $\phi = \pi/6$.

◇ See Figure 24. □

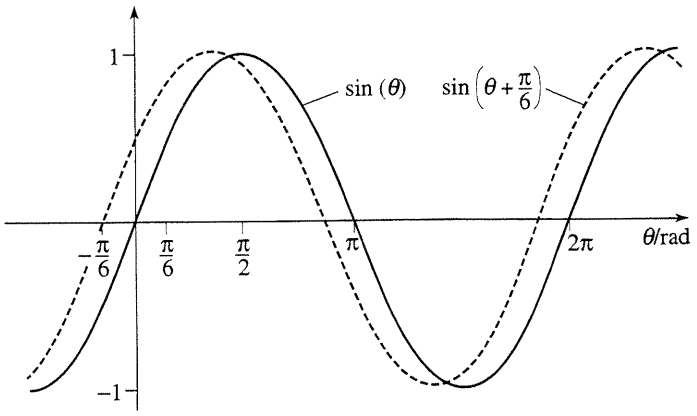


Figure 24 Graphs of sin (θ) and sin $(\theta + \pi/6)$.

Question T13 Use the terms periodic, symmetric or antisymmetric and sinusoidal, where appropriate, to describe the function tan (θ). What is the relationship between the graphs of tan (θ) and tan $(\phi + \theta)$? □

3.3 Inverse trigonometric functions

A problem that often arises in physics is that of finding an angle, θ, such that sin θ takes some particular numerical value. For example, given that sin $\theta = 0.5$, what is θ? You may know that the answer to this specific question is $\theta = 30°$ (i.e. $\pi/6$); but how would you write the answer to the general question, what is the angle θ such that sin $\theta = x$? The need to answer such questions leads us to define a set of **inverse trigonometric functions** that can 'undo' the effect of the trigonometric functions.

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1.1

These inverse functions are called **arcsine**, **arccosine** and **arctangent** (usually abbreviated to arcsin (x), arccos (x) and arctan (x)) and are defined so that:

arcsine
arccosine
arctangent

The abbreviations asin, acos and atan or alternatively $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, are sometimes used for the inverse trigonometric functions.

$$\arcsin(\sin\theta) = \theta \qquad -\pi/2 \le \theta \le \pi/2 \qquad (25a)$$

$$\arccos(\cos\theta) = \theta \qquad 0 \le \theta \le \pi \qquad (25b)$$

$$\arctan(\tan\theta) = \theta \qquad -\pi/2 < \theta < \pi/2 \qquad (25c)$$

Thus, since $\sin(\pi/6) = 0.5$, we can write $\arcsin(0.5) = \pi/6$ (i.e. 30°), and since $\tan(\pi/4) = 1$, we can write $\arctan(1) = \pi/4$ (i.e. 45°). Note that the *argument* of any inverse trigonometric function is just a number, whether we write it as x or $\sin\theta$ or whatever, but the *value* of the inverse trigonometric function is always an angle. Indeed, an expression such as $\arcsin(x)$ can be crudely read as 'the angle whose sine is x.'

Notice that Equations 25a–c involve some very precise restrictions on the values of θ, these are necessary to avoid ambiguity and deserve further discussion. Looking back at Figures 18, 19 and 20, you should be able to see that a single value of $\sin(\theta)$, $\cos(\theta)$ or $\tan(\theta)$ will correspond to an infinite number of different values of θ. For instance, $\sin(\theta) = 0.5$ corresponds to $\theta = \pi/6, 5\pi/6, 2\pi + (\pi/6), 2\pi + (5\pi/6)$, and any other value that may be obtained by adding an integer multiple of 2π to either of the first two values. To ensure that the inverse trigonometric functions are properly defined, we need to guarantee that each value of the function's argument gives rise to a *single* value of the function. The restrictions given in Equations 25a–c do ensure this, but they are a little too restrictive to allow those equations to be used as general definitions of the inverse trigonometric functions since they prevent us from attaching any meaning to an expression such as $\arcsin(\sin(7\pi/6))$.

In fact, the general definitions of the inverse trigonometric functions are as follows:

If $\sin(\theta) = x$, where $-\pi/2 \le \theta \le \pi/2$ and $-1 \le x \le 1$ (26a)

then $\arcsin(x) = \theta$

If $\cos(\theta) = x$, where $0 \le \theta \le \pi$ and $-1 \le x \le 1$ (26b)

then $\arccos(x) = \theta$

If $\tan(\theta) = x$, where $-\pi/2 < \theta < \pi/2$ (26c)

then $\arctan(x) = \theta$

In each case, the range of allowed x values constitutes the *domain* of the inverse function and the range of allowed θ values constitutes the *codomain*.

The graphs of these three functions are given in Figure 25.

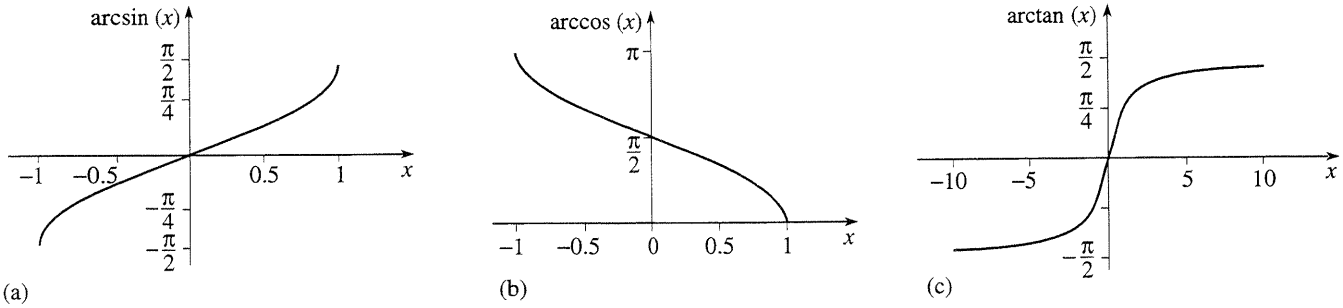


Figure 25 (a) The graph of $\arcsin(x)$. (b) The graph of $\arccos(x)$. (c) The graph of $\arctan(x)$.

Equations 26a–c look more intimidating than Equations 25a–c, but they embody the same ideas and they have the advantage of assigning meaning to expressions such as $\arcsin(\sin(7\pi/6))$.

◆ What is $\arcsin(\sin(7\pi/6))$?

◇ $\sin(7\pi/6) = -0.5$, so $\arcsin(\sin(7\pi/6)) = \arcsin(-0.5) = -\pi/6$. □

Inverse trigonometric functions can be found on all 'scientific' calculators, often by using the trigonometric function keys in combination with the inverse key. The

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answer will be given in either radians or degrees, depending on the mode selected on the calculator, and will always be in the standard angular ranges given in Equations 26a–c.

When dealing with functions of any kind, physicists traditionally pay scant regard to such mathematical niceties as *domains* (ranges of allowed argument values) and *codomains* (ranges of allowed function values). This attitude persists because it rarely leads to error. However, care *is* needed when dealing with inverse trigonometric functions. If you know the sin, cos or tan of an angle and you want to find the angle itself, the inverse trigonometric functions will give you *an* answer. But that answer will lie in one of the standard angular ranges and may not be the particular answer you are seeking. You will have to remember that there are other possible answers, your calculator is unlikely to warn you!

◆ Earlier, arcsin (x) was crudely interpreted as 'the angle with a sine of x.' What would be a more accurate interpretation?

◇ 'The angle in the range $-\pi/2$ to $\pi/2$ with a sine of x.' □

Question T14 (a) Use a calculator to find arcsin (0.65) both in radians and in degrees.

(b) Using information from earlier in this module, determine the values of:

$\arctan(1)$ $\arcsin\left(\sqrt{3}/2\right)$ $\arccos\left(1/\sqrt{2}\right)$ □

It is possible to define a set of **inverse reciprocal trigonometric functions** in much the same way that we defined the inverse trigonometric functions. These functions are called **arccosecant**, **arcsecant** and **arccotangent**, and are usually abbreviated to arccosec, arcsec and arccot. The definitions are given below and the corresponding graphs are given in Figure 26.

inverse reciprocal trigonometric functions
arccosecant
arcsecant
arccotangent

If $\operatorname{cosec}(\theta) = x$

where either $0 < \theta \leq \pi/2$ and $x \geq 1$

or $-\pi/2 \leq \theta < 0$ and $x \leq -1$

then $\operatorname{arccosec}(x) = \theta$

If $\sec(\theta) = x$

where either $0 \leq \theta < \pi/2$ and $x \geq 1$

or $\pi/2 < \theta \leq \pi$ and $x \leq -1$

then $\operatorname{arcsec}(x) = \theta$

If $\cot(\theta) = x$

where $-\pi/2 < \theta < \pi/2$ and $-\infty < x < +\infty$

then $\operatorname{arccot}(x) = \theta$

The codomains (i.e. ranges of θ) quoted here are not fully standardized. For example, some algebraic computing packages take the codomain of arctan (x) to be $0 < \theta < \pi$.

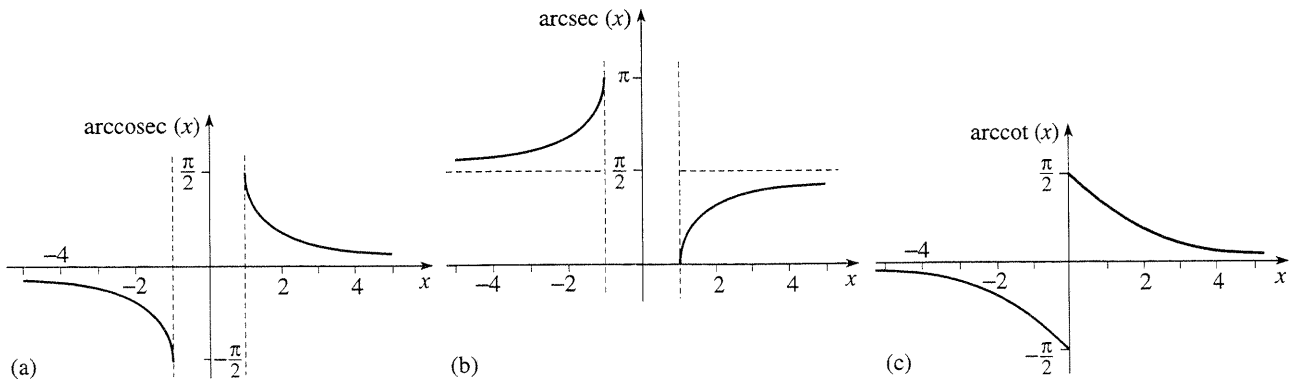


Figure 26 The graphs of (a) arccosec (x), (b) arcsec (x), and (c) arccot (x).

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Finally, a few words of warning. The following notation is sometimes used to represent the inverse trigonometric functions:

$\sin^{-1}(x)$ for arcsin (x)

$\cos^{-1}(x)$ for arccos (x)

$\tan^{-1}(x)$ for arctan (x)

Notice that there is *no* connection with the *positive* index notation used to denote powers of the trigonometric functions (for example, using $\sin^2(\theta)$ to represent $(\sin(\theta))^2$. Also notice that although this notation might make it appear otherwise, there is still a clear distinction between the *inverse trigonometric functions* and the *reciprocal trigonometric functions*.

So $\sin^{-1}[\sin(\theta)] = \theta$

but $\text{cosec}(\theta) \times \sin(\theta) = 1$

Question T15 Figure 27 shows a ray of light travelling from glass to water. The angles, θ_i and θ_r are related by Snell's law:

$$\mu_g \sin\theta_i = \mu_w \sin\theta_r$$

where μ_g and μ_w are constants.

(a) If the measured values of the angles are $\theta_i = 50.34°$ and $\theta_r = 60.00°$, what is the value of μ_g/μ_w?

(b) The *critical angle*, θ_c, is defined as the value of θ_i for which θ_r is 90°. Use your result for μ_g/μ_w to determine the critical angle for this glass–water interface. □

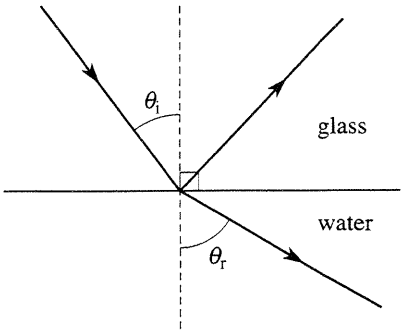


Figure 27 See Question T15.

4 Trigonometric rules and identities

In this section we first extend the discussion in Section 2 to derive relationships between sides and angles of any arbitrary triangle, and then add to the discussion of Subsections 2.3 and 3.2 by introducing some further relationships involving trigonometric functions.

Study comment It is unlikely that you will be required to reproduce the derivations given in this section. However, you should try to ensure that you understand each step, since that will reinforce your understanding and experience of trigonometric functions.

4.1 The sine and cosine rules

In Section 2, you saw that the sides and angles of a right-angled triangle are related by Pythagoras's theorem and by the trigonometric ratios. These relationships are in fact special cases of relationships that apply to *any* triangle. To derive these general relationships, consider Figure 28, where ABC is any triangle, with sides a, b and c. BD is drawn perpendicular to AC and has length p. We have called the length of AD, q, which means that the length of DC is $b-q$.

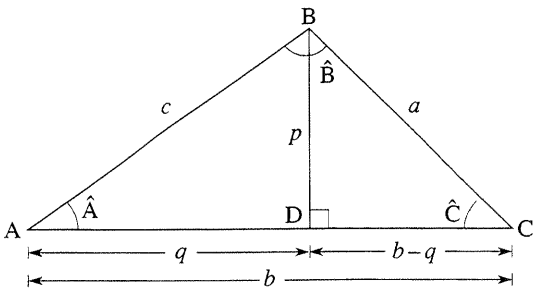


Figure 28 Construction used in deriving the sine and cosine rules.

From Figure 28,

$$\sin\hat{A} = \frac{p}{c}, \text{ so } p = c\sin\hat{A}$$

$$\sin\hat{C} = \frac{p}{a}, \text{ so } p = a\sin\hat{C}$$

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and therefore $a \sin \hat{C} = c \sin \hat{A}$, or

$$\frac{a}{\sin \hat{A}} = \frac{c}{\sin \hat{C}}$$

Similarly, by drawing a perpendicular from C to AB,

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}}$$

By combining the two previous equations we obtain

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} \qquad (27)$$

which is called the **sine rule**.

sine rule
cosine rule
identity
trigonometric identities

◆ What is the length, x, in Figure 29?

◇ The remaining angle is $180° - 2 \times 50° = 80°$, hence

$$\frac{x}{\sin(50°)} = \frac{1\,\text{m}}{\sin(80°)}$$

and $$x = \frac{1\,\text{m} \times \sin(50°)}{\sin(80°)} \approx 0.778\ \text{m}$$ □

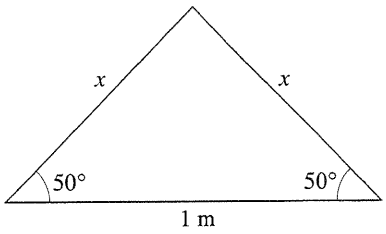


Figure 29 A triangle with two equal sides of unknown length x.

There is also an important relationship between the three sides of a general triangle and the cosine of one of the angles. To derive this rule, consider Figure 28 again. ABD is a right-angled triangle so Pythagoras's theorem gives

$$c^2 = q^2 + p^2$$

In a similar way, since BCD is a right-angled triangle, we have:

$$a^2 = p^2 + (b - q)^2$$

Eliminating p^2 between these two *simultaneous equations* gives us:

$$a^2 = (c^2 - q^2) + (b - q)^2$$

which we can simplify by expanding the bracket on the right-hand side:

$$a^2 = c^2 - q^2 + b^2 - 2bq + q^2 = c^2 + b^2 - 2bq$$

But $q = c \cos \hat{A}$ and so:

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A} \qquad (28)$$

which is called the **cosine rule**.

It is possible for the value of one of the angles in a general triangle to be greater than 90° — the sine rule is still valid for such cases, since the trigonometric *function* sin $(\hat{A})$, for example, can be used instead of the trigonometric *ratio* sin $\hat{A}$.

Notice that the cosine rule reduces to Pythagoras's theorem for $\hat{A} = 90°$.

The cosine rule is also valid for $90° < \hat{A} < 180°$.

Question T16 Figure 30 shows the path of a ship that sailed 30 km due east, then turned through 120° and sailed a further 40 km. Calculate its distance x 'as the crow flies' from the starting point. □

Question T17 One of the interior angles of a triangle is 120°. If the sides adjacent to this angle are of length 4 m and 5 m, use the cosine rule to find the length of the side opposite the given angle. □

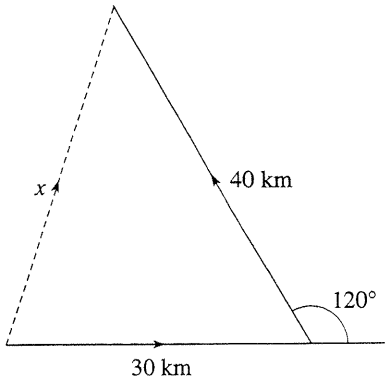


Figure 30 See Question T16.

4.2 Trigonometric identities

A good deal of applicable mathematics is concerned with equations. It is generally the case that these equations are only true when the variables they contain take on certain specific values; for example, $4x = 4$ is only true when $x = 1$. However, we sometimes write down equations that are true for *all* values of the variables, such as $(x + 1)^2 = x^2 + 2x + 1$. Equations of this latter type, i.e. ones that are true irrespective of the specific values of the variables they contain, are properly called **identities**. (☞)

There are a great many **trigonometric identities**, i.e. relationships between trigonometric functions that are independent of the specific values of the variables they involve. These have various applications and it is useful to have a list of them

To show that two expressions are related by an identity we sometimes use the symbol ≡ which should be read as 'is identical to' rather than the more familiar = which should be read as 'is equal to.'

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for easy reference. The most important are given below — you have already met the first seven (in slightly different forms) earlier in the module and others occur at various points throughout *FLAP*. Note that α and β may represent *any* numbers or angular values, unless their values are restricted by the definitions of the functions concerned.

symmetry relations
basic identities
addition formulae
double angle formulae
half-angle formulae

The **symmetry relations**: (☞)

The symmetry relations show the oddness or evenness of the functions.

$$\sin(-\alpha) = -\sin(\alpha) \quad \text{(Eqn 18)}$$

$$\cos(-\alpha) = \cos(\alpha) \quad \text{(Eqn 19)}$$

$$\tan(-\alpha) = -\tan(\alpha) \quad \text{(Eqn 20)}$$

The **basic identities**: (☞)

The basic identities arise from basic definitions and Pythagoras's theorem.

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} \quad \text{(Eqn 8)}$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \quad \text{(Eqn 9)}$$

$$\cot(\alpha) = \frac{\operatorname{cosec}(\alpha)}{\sec(\alpha)} \quad \text{(Eqn 16)}$$

$$1 + \tan^2(\alpha) = \sec^2(\alpha) \quad \alpha \neq (2n+1)\frac{\pi}{2} \quad \text{(Eqn 17)}$$

$$\cot^2(\alpha) + 1 = \operatorname{cosec}^2(\alpha)$$

The **addition formulae**: (☞)

Replacing β by $-\beta$ and using the symmetry relations to replace $\sin(-\beta)$ by $-\sin(\beta)$, $\cos(-\beta)$ by $\cos(\beta)$, and $\tan(-\beta)$ by $-\tan(\beta)$ leads to further addition formulae (sometimes called the *subtraction formulae*) for $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha-\beta)$.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad (29)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad (30)$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \quad (31)$$

for α, β and $(\alpha+\beta)$ not equal to $\frac{(2n+1)\pi}{2}$

The **double-angle formulae**: (☞)

The double-angle formulae come from the addition formulae with $\alpha = \beta$. When deriving Equations 34 and 35, it is also necessary to make use of Equation 9.

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \quad (32)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (33)$$

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) \quad (34)$$

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1 \quad (35)$$

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{[1-\tan^2(\alpha)]} \quad (36)$$

for α and 2α not equal to $\frac{(2n+1)\pi}{2}$

The **half-angle formulae**: (☞)

Equations 37 and 38 follow from Equations 34 and 35.

$$\cos^2(\alpha/2) = \tfrac{1}{2}[1+\cos(\alpha)] \quad (37)$$

$$\sin^2(\alpha/2) = \tfrac{1}{2}[1-\cos(\alpha)] \quad (38)$$

and if $t = \tan(\alpha/2)$ then, provided $\tan(\alpha/2)$ is defined:

$$\sin(\alpha) = \frac{2t}{1+t^2} \quad (39)$$

$$\cos(\alpha) = \frac{1-t^2}{1+t^2} \quad (40)$$

$$\tan(\alpha) = \frac{2t}{1-t^2} \quad \text{for } \alpha \text{ not equal to } \frac{(2n+1)\pi}{2} \quad (41)$$

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The **sum formulae**: (☞)

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad (42)$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \quad (43)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad (44)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \quad (45)$$

sum formulae
product formulae

The sum formulae are used in the study of *oscillations* and waves, where we often need to combine two oscillations to determine their combined effect.

The **product formulae**: (☞)

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad (46)$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \quad (47)$$

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta) \quad (48)$$

$$-2\sin(\alpha)\sin(\beta) = \cos(\alpha+\beta) - \cos(\alpha-\beta) \quad (49)$$

The product formulae can be of value in the procedure known as *integration*.

As indicated by the marginal notes, some of these identities can be derived straightforwardly from others. Rather than deriving the unexplained identities, one by one, we will consider just one example in full and allow it to serve as an example.

To derive Equation 29, the addition formula for $\sin(\alpha+\beta)$, we use Figure 31. The line OB is at an angle, α, to the x-axis. OA has unit length and is at an angle, β, to OB. The angle $A\hat{B}O$ is a right angle. (This is what fixes the length of OB, which is *not* of unit length.) It turns out to be useful to construct lines AN, BM and TB, such that $A\hat{N}O$, $B\hat{M}O$ and $B\hat{T}A$, are all right angles. If we let S represent the point of intersection of OB and AN, then $O\hat{S}N$, and consequently $A\hat{S}B$, are equal to $\alpha - 90°$. But SAB is a right-angled triangle, so $T\hat{A}B$ is equal to α.

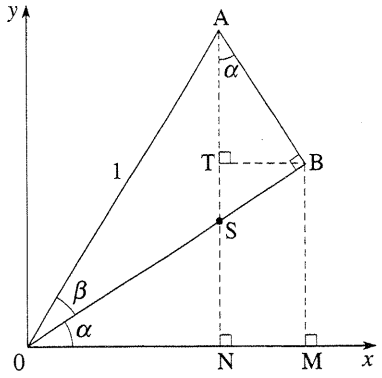


Figure 31 Construction used in deriving Equation 29.

Since OA has unit length, the y-coordinate of the point, A, is $\sin(\alpha+\beta)$ and from the diagram we have:

$$\sin(\alpha+\beta) = NT + TA \quad (50)$$

The second addition formula, Equation 30, can be derived in a similar way, by considering the x-coordinate of the point, A.

To find TA, first consider the triangle, OAB. Since OA = 1, we have:

$$AB = \sin(\beta) \quad (51)$$

Next, considering triangle TAB, we have

$$\cos(\alpha) = \frac{TA}{AB} = \frac{TA}{\sin(\beta)} \quad (52)$$

Therefore,

$$TA = \cos(\alpha)\sin(\beta) \quad (53)$$

To find NT, we start by considering triangle OAB, obtaining

$$OB = \cos(\beta) \quad (54)$$

Next, consideration of triangle OBM gives

$$\sin(\alpha) = \frac{MB}{OB} = \frac{NT}{\cos(\beta)} \quad (55)$$

Therefore

$$NT = \sin(\alpha)\cos(\beta) \quad (56)$$

The derivation given here assumes that $0 < (\alpha+\beta) < \pi/2$, but Equations 29 and 30 can be shown to be valid for all values of α and β.

We can now substitute in Equation 50 using Equations 53 and 56 to get the desired result:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad \text{(Eqn 29)}$$

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The first of the double-angle identities, Equation 32, can be obtained by putting $\beta = \alpha$ in Equation 29:

$$\sin(\alpha + \alpha) = 2 \sin(\alpha) \cos(\alpha)$$

◆ Use the addition identities and values from Table 3 to calculate the exact value of sin 75°.

◇ From Equation 29, $\sin 75° = \sin(45° + 30°)$

$$= \sin 45° \cos 30° + \cos 45° \sin 30° \quad (☞)$$

Using Table 4 (see Answer T5).

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} \quad \square$$

Question T18 Derive the identity given in Equation 33. □

Question T19 Use Equation 34 to find the exact value of sin (15°). □

Question T20 A feather trapped on the front of a loudspeaker moves back and forth whenever the speaker is in use. When a certain signal is played through the speaker, the position of the feather at time t is given by $x(t) = A \cos(\omega t + \phi_1)$, where A, ω and ϕ_1 are constants. A second similar signal causes a motion described by $x(t) = A \cos(\omega t + \phi_2)$, where ϕ_2 is another constant. Using an appropriate trigonometric identity, find a compact expression describing the motion $x(t) = A \cos(\omega t + \phi_1) + A \cos(\omega t + \phi_2)$ that results from relaying the two signals simultaneously. □

5 Closing items

5.1 Module summary

1 Angles are usually measured in *degrees* or *radians*. An angle of 180° is equivalent to π rad.

2 The side opposite the *right angle* in a *right-angled triangle* is called the *hypotenuse*.

3 *Pythagoras's theorem* states that the square of the hypotenuse in a right-angled triangle is the sum of the squares of the other two sides. (The converse is also true.)

4 For a right-angled triangle, the *trigonometric ratios* are:

Useful mnemonics are s*oh*, c*ah* and t*oa*.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{(Eqn 5)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{(Eqn 6)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{(Eqn 7)}$$

5 The *reciprocal trigonometric ratios* are:

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \quad \text{provided } \sin\theta \neq 0 \qquad \text{(Eqn 10)}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{provided } \cos\theta \neq 0 \qquad \text{(Eqn 11)}$$

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{provided } \tan\theta \neq 0 \qquad \text{(Eqn 12)}$$

6 For a small angle θ, $\cos\theta \approx 1$

and $\sin\theta \approx \tan\theta \approx \theta/\text{rad}$

7 The sides and angles of a general triangle are related by the *sine rule*:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}} \qquad \text{(Eqn 27)}$$

and the *cosine rule*:

$$a^2 = b^2 + c^2 - 2bc\cos\hat{A} \qquad \text{(Eqn 28)}$$

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8 The *trigonometric functions* generalize the trigonometric ratios; $\sin(\theta)$ and $\cos(\theta)$ are *periodic functions* (with *period* 2π) and are defined for any value of θ. The function $\tan(\theta)$ is a periodic function (with period π) and is defined for all values of θ except odd multiples of $\pi/2$. The *reciprocal trigonometric functions* generalize the reciprocal trigonometric ratios in a similar way.

9 The *inverse trigonometric functions*; $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ are defined as follows:

$$\text{If } \sin(\theta) = x, \quad \text{where } -\pi/2 \leq \theta \leq \pi/2 \text{ and } -1 \leq x \leq 1 \tag{26a}$$

$$\text{then} \quad \arcsin(x) = \theta$$

$$\text{If } \cos(\theta) = x, \quad \text{where } 0 \leq \theta \leq \pi \text{ and } -1 \leq x \leq 1 \tag{26b}$$

$$\text{then} \quad \arccos(x) = \theta$$

$$\text{If } \tan(\theta) = x, \quad \text{where } -\pi/2 < \theta < \pi/2 \tag{26c}$$

$$\text{then} \quad \arctan(x) = \theta$$

The *inverse reciprocal trigonometric functions* may be defined in a similar way.

10 There are a large number of *trigonometric identities*, such as

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{Eqn 9}$$

the most important are listed in Subsection 4.2 of this module.

5.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Given an angle measured in degrees express its value in radians, and vice versa.

A3 Use Pythagoras's theorem.

A4 Use trigonometric ratios, and their reciprocals, to solve geometrical problems.

A5 Explain the sine and cosine rules and use these rules to solve problems involving general triangles.

A6 Explain how the sine, cosine and tangent functions can be defined for general angles and sketch their graphs.

A7 Recognize the most common identities for the trigonometric functions and apply them to solving mathematical and physical problems.

A8 Use the inverse trigonometric functions to solve mathematical and physical problems.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2*) Define the units commonly used in the measurement of angles.

Express the angles, $-10°$, $0°$, $275°$, in terms of radians.

Write the angles $2\pi/3$ and $\pi/4$ in terms of degrees.

Question E2 (*A3*) State Pythagoras's theorem.

If the two shorter sides of a right-angled triangle have lengths of 4961 m and 6480 m, what is the length of the third side?

Question E3 (*A4*) By drawing a suitable diagram, give definitions of the sine, cosine and tangent ratios. A right-angled triangle has a hypotenuse of length 7 mm, and one angle of $55°$. Determine the lengths of the other two sides.

Question E4 (*A5*) Two sides of a triangle have lengths 2 m and 3 m. If the angle between these two sides is $60°$, find the length of the third side and find the remaining angles.

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Question E5 (*A6*) Sketch the graphs of sin (θ), cos (θ) and tan (θ) for general values of θ. Why is tan (θ) undefined when θ is an odd multiple of $\pi/2$?

Question E6 (*A7*) Use Figure 10 and suitable trigonometric identities to show that:

$$\sin(22.5°) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

Question E7 (*A7*) Given that $\sin(60°) = \sqrt{3}/2$ and $\cos(60°) = 1/2$, use suitable trigonometric identities to find the exact values of sin (120°), cos (120°) and cot ($2\pi/3$) without using a calculator.

Question E8 (*A8*) Determine a general expression for all values of θ that satisfy the equation:

$$\cot(\theta) = 4/3$$

(*Hint*: Solve for $-\pi/2 < \theta < \pi/2$ and then consider how tan (θ), and hence cot (θ), behaves for other angles.)

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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6 Answers and comments

Fast track answers

F1 Pythagoras's theorem states that the square of the hypotenuse in a right-angled triangle is equal to the sum of the squares of the other two sides.

F2 With the notation of Figure 8 (Subsection 2.2), we have:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

From Pythagoras's theorem, the hypotenuse of the 45° triangle in Figure 10 (Subsection 2.2) has length $\sqrt{2}$, and therefore:

$$\sin(45°) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

An angle of $\pi/4$ radians is equivalent to 45° and therefore we have:

$$\tan\left(\frac{\pi}{4}\right) = 1$$

F3 The sine rule $\quad \dfrac{a}{\sin\hat{A}} = \dfrac{b}{\sin\hat{B}} = \dfrac{c}{\sin\hat{C}}$

The cosine rule $\quad a^2 = b^2 + c^2 - 2bc\cos\hat{A}$

where the notation is given in Figure 28 (Subsection 4.1).

Suppose $b = c = 2$ m and $a = 3$ m. Then using the cosine rule we get:

$$3^2 = 2^2 + 2^2 - 2\times 2\times 2\cos\hat{A}$$

and so:

$$\cos\hat{A} = -1/8$$

There are an infinite number of solutions to this equation for $\hat{A}$, but only one is relevant to our triangle problem:

$$\hat{A} = \arccos(1/8) = 97.2°$$

Since sides b and c of the triangle are equal, we have:

$$\hat{B} = \hat{C} = \frac{180° - 97.2°}{2} = 41.4°$$

F4 The graphs for these functions are given in Figures 21, 22 and 23 of the module.

F5 The graphs for these functions are given in Figures 26a and 26c of the module.

Ready to study answers

R1 (a) $1/x = h/a$, (b) $x/y = (a/h)/(b/h) = a/b$,

(c) $x^2 + y^2 = a^2/h^2 + b^2/h^2 = (a^2 + b^2)/h^2 = 1$.

(If you had difficulty with any part of this question, consult *algebra* and *arithmetic* in the *Glossary*.)

R2 Multiplying the first of the given equations by 2, the pair become

$$4x + 10y = 12 \text{ and } 3x - 10y = 9$$

Adding the corresponding sides of the two equations in order to *eliminate* y, we find

$$7x = 21, \text{ so } x = 3$$

Substituting $x = 3$ into either of the original equations shows that $y = 0$.

(If you had difficulty with any part of this question, consult *solving simultaneous equations* in the *Glossary*.)

R3 A function is a rule that assigns a single value from a set called the *codomain* to each value from a set called the *domain*. Thus, saying that the position of an object is a function of time implies that at each instant of time the object has one and only one position.

(If you had difficulty with any part of this question, consult *functions* in the *Glossary*.)

Text answers

T1 An angle of π radians is equivalent to 180°, so 1 radian is equivalent to $180/\pi \approx 57.3°$. This means that the conversion factor from radians to degrees is (to three significant figures) 57.3 degrees radian^{-1}. Similarly, an angle of 1° is equivalent to $(\pi/180)$ rad = 0.0175 rad and so the conversion factor from degrees to radians is approximately 0.0175 radians degree^{-1}.

T2 First find the number of times 2π rad divides into 201π rad:

$$\frac{201\pi}{2\pi} = 100.5$$

So, in order to convert to the given range we need to add $101\times 2\pi$ rad to the given angle. Thus, -201π rad reduces to -201π rad + $101\times 2\pi$ rad = π rad.

T3 In both cases we calculate $\sqrt{b^2 + c^2}$ where b and c are the two shorter sides, obtaining:

$$\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

and

$$\sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$

So both triangles are right-angled, by the converse of Pythagoras's theorem.

T4 If the hypotenuse of a right-angled triangle is labelled, a, and the other two sides are b and c, then:

$$a^2 = b^2 + c^2$$

where a, b and c are positive. Since $c^2 > 0$ we have:

$a^2 > b^2$ and consequently $a > b$.

Also, $b^2 > 0$, and so we obtain:

$a^2 > c^2$ and consequently $a > c$.

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T5 By Pythagoras's theorem, the perpendicular has length, $\sqrt{2^2-1} = \sqrt{3}$. Therefore the completed table is as given in Table 4.

Table 4 See Answer T5.

θ /degrees	θ /radians	sin θ	cos θ	tan θ
0	0	0	1	0
30	$\pi/6$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
45	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90	$\pi/2$	1	0	undefined

T6 Graphs of (a) cos θ and (b) tan θ for $0 \le \theta < \pi/2$ are given in Figure 32.

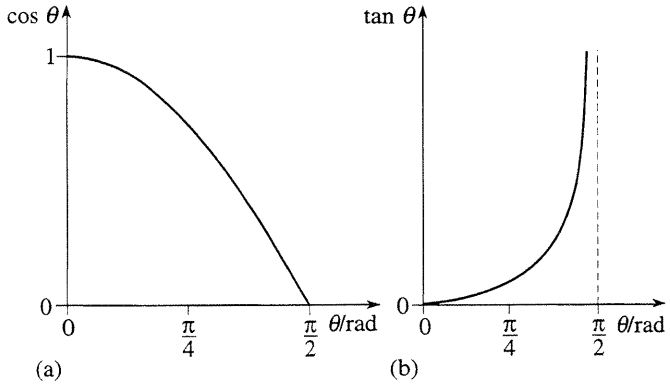


Figure 32 See Answer T6.

T7 The angle $A\hat{B}C = 90° - \theta$, so the angle $B\hat{A}C$ must be θ. Since the side AB is the hypotenuse of the right-angled triangle ABC, it follows that

$$\sin\theta = \lambda/d$$

and $\lambda = 2 \times 10^{-6}\,\text{m} \times \sin(17.46°) = 6 \times 10^{-7}\,\text{m}$

T8 From Figure 11:

cosec (30°) = 2, sec (30°) = $2/\sqrt{3}$, cot (30°) = $\sqrt{3}$

cosec (60°) = $2/\sqrt{3}$, sec (60°) = 2, cot (60°) = $1/\sqrt{3}$

T9 cosec (23°) = 1/sin (23°) = 2.559;
sec (56°) = 1/cos (56°) = 1.788;
cot (π/6) = 1/tan (π/6) = 1.732;
cot (1.5) = 1/tan (1.5) = 0.071.

T10 Graphs of (a) sec θ (b) cot θ for $0 \le \theta < \pi/2$ are given in Figure 33.

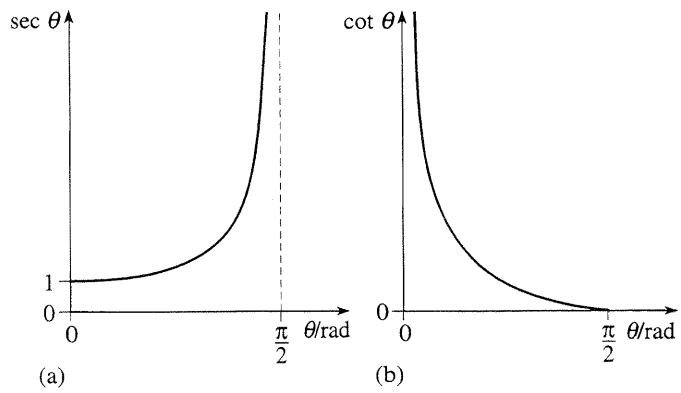


Figure 33 See Answer T10.

T11 $\phi = 0.5° = (0.5 \times \pi/180)$ rad $= (0.5 \times 0.0175)$ rad $= 8.73 \times 10^{-3}$ rad (see Answer T1 for the origin of the conversion factor).

Since ϕ is a small angle, $\phi/\text{rad} \approx s/d$ so $s \approx d \times \phi/\text{rad} = d \times 8.73 \times 10^{-3}$.

T12 The sin (θ) graph repeats itself so that sin ($2\pi + \theta$) = sin (θ). It is antisymmetric, i.e. sin (θ) = $-$ sin ($-\theta$) and continuous, and any value of θ gives a unique value of sin (θ).

The cos (θ) graph repeats itself so that cos ($2\pi + \theta$) = cos (θ). It is symmetric, i.e. cos (θ) = cos ($-\theta$) and continuous, and any value of θ gives a unique value of cos (θ).

T13 The function tan (θ) is antisymmetric, that is tan (θ) = $-$tan ($-\theta$); it is periodic with period π; it is *not* sinusoidal. The graph of tan ($\theta + \phi$) has the same shape as that of tan (θ), but is shifted to the left by ϕ.

T14 (a) With a calculator in degree mode, arcsin (0.65) = 40.5°, and in radian mode, arcsin (0.65) = 0.708 rad.

(b) Using Table 3 and/or Figures 10 and 11: tan (π/4) = 1, so arctan (1) = π/4 (i.e. 45°); sin (π/3) = $\sqrt{3}/2$ and so arcsin ($\sqrt{3}/2$) = π/3 (i.e. 60°); cos (π/4) = $1/\sqrt{2}$, therefore arccos ($1/\sqrt{2}$) = π/4 (i.e. 45°).

T15 (a) Using Snell's law we find:

$$\frac{\mu_g}{\mu_w} = \frac{\sin\theta_r}{\sin\theta_i} = \frac{\sin(60°)}{\sin(50.34°)} = 1.124\,934 = 1.12$$ (to two decimal places)

(b) Again using Snell's law, we have:

$$\sin\theta_i = \frac{\sin(90°)}{1.124\,934}$$

and so

$$\theta_c = \arcsin\left(\frac{1}{1.124\,934}\right) = 62.7°$$ (to one decimal place)

Notice that if you use the approximate value $\mu_g/\mu_w = 1.12$ in (b) you obtain sin θ_i = 1/1.12 and θ_c = 63.2°. To avoid such *rounding errors*, you should always work to a higher degree of precision (i.e. keep more figures) at intermediate stages than will be justified in your final answer, and round to a sensible number of figures only at the last step. Also notice that, although there are an infinite number of solutions to the equation for θ_c, only the one given is physically admissible.

T16 Using the cosine rule (Equation 28):

$$x^2 = (40\text{ km})^2 + (30\text{ km})^2 - 2 \times 40\text{ km} \times 30\text{ km} \times \cos(180° - 120°)$$
$$= [2500\text{ km}^2 - 2400\cos(60°)]\text{ km}^2 = 1300\text{ km}^2$$

and hence: $x = \sqrt{1300}$ km = 36.1 km

T17 Using the cosine rule (Equation 28):

$$x^2 = (4\text{ m})^2 + (5\text{ m})^2 - 2 \times 4\text{ m} \times 5\text{ m} \times \cos(120°)$$
$$= 41\text{ m}^2 - 40\cos(120°)\text{ m}^2 = 41\text{ m}^2 + (40/2)\text{ m}^2 = 61\text{ m}^2$$

and so: $x = \sqrt{61}$ m = 7.81 m

T18 Setting $\alpha = \beta$ in Equation 30 we get:

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

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T19 Let $\alpha = 15°$ then, from Equation 34:

$$\cos(30°) = 1 - 2\sin^2(15°) = \frac{\sqrt{3}}{2}$$

so that $$2\sin^2(15°) = \frac{2-\sqrt{3}}{2}$$

and therefore $$\sin(15°) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

T20 Using Equation 44 with $\alpha = \omega t + \phi_1$ and $\beta = \omega t + \phi_2$, we find

$$x(t) = 2A\cos\left(\frac{\omega t + \phi_1 - \omega t - \phi_2}{2}\right)\cos\left(\frac{\omega t + \phi_1 + \omega t + \phi_2}{2}\right)$$

so $$x(t) = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\cos\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

Exit test answers

E1 A degree is defined as 1/360th of a rotation through a complete circle. A radian is the angle subtended at the centre of a circle by an arc of the circle that has an arc length equal to the radius of the circle. Since π radians is equivalent to 180°, we have:

$-10°$ is equivalent to $-(10\pi/180)$ radians = -0.175 radians

0° is equivalent to 0 radians

275° is equivalent to $(275\pi/180)$ radians = 4.8 radians

$2\pi/3$ radians is equivalent to $2\pi/3 \times 180°/\pi = 120°$

$\pi/4$ radians is equivalent to $\pi/4 \times 180°/\pi = 45°$.

(Reread Subsection 2.1 if you had difficulty with this question.)

E2 Pythagoras's theorem states that the square of the hypotenuse in a right-angled triangle is the sum of the squares of the other two sides. If we call the hypotenuse x, then:

$$x = \sqrt{(4961\,\text{m})^2 + (6480\,\text{m})^2} = 8161\,\text{m}$$

(Reread Subsection 2.2 if you had difficulty with this question.)

E3 Using the conventional notation (see Figure 8) the definitions are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The side opposite the 55° angle has length 7.0 mm × sin (55°) = 5.7 mm. The side adjacent to the 55° angle has length 7.0 mm × cos (55°) = 4.0 mm.

(Reread Subsection 2.3 if you had difficulty with this question.)

E4 Using the cosine rule (Equation 28) with b = 2 m, c = 3 m and $\hat{A} = 60°$, we find:

$$a^2 = 2^2 + 3^3 - 2\times 2\times 3\times \tfrac{1}{2} = 7$$

and therefore $a = \sqrt{7}$ m.

The sine rule (Eqn 27) gives: $$\frac{\sqrt{7}}{\sin(60°)} = \frac{2}{\sin(\hat{B})}$$

and hence: $$\sin(\hat{B}) = \frac{2}{\sqrt{7}} \times \frac{\sqrt{3}}{2} = \sqrt{\frac{3}{7}}$$

which implies that: $$\hat{B} = 40.9°$$

Moreover, since the angles of a triangle add up to 180°, we also find: $\hat{C} = 180 - 60 - 40.9 = 79.1°$.

(Reread Subsection 4.1 if you had difficulty with this question.)

E5 See Figures 18, 19 and 20. From Equation 8, the function tan (θ) becomes infinite whenever cos (θ) = 0, that is whenever $\theta = (2n + 1)\pi/2$ for any integer, n. Looking at Figure 20, we see that approaching such values of θ from below, tan (θ) tends to $+\infty$ and approaching from above, tan (θ) tends to $-\infty$. Consequently, tan $[(2n+1)\pi/2]$ is undefined.

(Reread Subsections 3.1 and 3.2 if you had difficulty with this question.)

E6 From Figure 10 we obtain: $\cos(45°) = 1/\sqrt{2}$

We can rewrite Equation 34 as: $$\sin^2(a) = \frac{1-\cos(2\alpha)}{2}$$

which gives: $$\sin^2(22.5°) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Taking the positive square root (since we know that sin (θ) is positive for $0° < \theta < 90°$), we obtain:

$$\sin(22.5°) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

(Reread Subsection 4.2 if you had difficulty with this question.)

E7 Using Equation 32 we have:

$$\sin(120°) = 2\sin(60°)\cos(60°) = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$$

Furthermore, using Equation 34 we have:

$$\cos(120°) = 1 - 2\sin^2(60°) = 1 - 2\times\frac{3}{4} = -\frac{1}{2}$$

But $2\pi/3$ radians is equivalent to $2\pi/3 \times 180/\pi = 120°$, so:

$$\cot\left(\frac{2\pi}{3}\right) = \frac{\cos(120°)}{\sin(120°)} = -\frac{1}{2} \times \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

(Reread Subsection 4.2 if you had difficulty with this question.)

E8 Since cot (θ) = 1/tan (θ), tan θ = 3/4 = 0.75. With a calculator set in radians mode, arctan (0.75) = 0.644. But tan (θ) is periodic, with a period π, so the complete solution is $\theta = 0.644 + n\pi$, where n is any integer.

In degree mode, arctan (0.75) = 36.9°, and the complete solution is $\theta = 36.9° + 180°\,n$.

(Reread Subsections 3.2 and 3.3 if you had difficulty with this question.)

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