Exponential and logarithmic functions

Module M1.5

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1 Opening items

1.1 Module introduction

When the electric charge stored in a capacitor is discharged through a resistor, the rate of flow of charge through the resistor is proportional to the charge remaining on the capacitor. In a population of breeding organisms, the number of offspring produced in a given time, and hence the rate of population growth, is proportional to the size of the population. These processes of electrical discharge and population growth both provide examples of *exponential change*.

Exponential changes are the subject of Section 2 of this module. Subsection 2.1 introduces some more examples of exponential change and uncovers some of their common characteristics. Subsection 2.2 concerns the *rate of change* of a quantity and shows how this can be related to the gradient of the *tangent* to the graph of that quantity. In particular, by requiring that the rate of change of a quantity should always be equal to the instantaneous value of the quantity itself, we are led to define an *exponential function*, of the form $y(x) = e^x$, where e is an important mathematical constant, equal to 2.718 (to three decimal places). Subsection 2.3 examines the general mathematical properties of exponential functions, and in Subsection 2.4, exponential functions are used to describe various examples of exponential change, including the decay of radioactive nuclei. Section 2 ends with a more mathematical approach to the definition and evaluation of the number e that involves the concept of a *limit*.

In Section 3, *logarithmic functions* (logs) are introduced. We see how logarithms can be expressed in different *bases*, how the logs of products, quotients and powers can be expanded, and how the base of a logarithm can be changed. The *antilog* function is also introduced, and we look at how logs, antilogs and exponential functions can be handled on a calculator. The module ends with a brief look at how logarithmic functions are used in physics to analyse data that obey an exponential law or a *power law*.

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions?* The answers are given in Section 5. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than one or two of the questions you are strongly advised to study the whole module.*

Question F1 Plot a graph of $y = x^2$ by evaluating y when $x = 0, \pm1, \pm2, \pm3$. Estimate the gradient (i.e. slope) at $x = 2$ by drawing a tangent to the curve.

Question F2 Explain what is meant by:

$$\lim_{x\to\infty} \frac{1}{x}$$

Question F3 State the usual symbol for the following expression, and give its value to three decimal places:

$$\lim_{m\to\infty} (1 + 1/m)^m$$

Question F4 What is the gradient of the graph of $y = \exp(kx)$ at $x = 0$?

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Question F5 Where possible, simplify the following expressions:

(a) $\log_e[(e^x)^y]$

(b) $\log_e(e^x + e^{2y})$

(c) $\exp[\log_e(x) + 2\log_e(y)]$

(d) $\exp[2\log_e(x)]$

(e) $a^{\log_a(x)}$

Question F6 If $P = kf^{-a}$, what are the gradient, and the intercept on the vertical axis, of the graph of $\log_{10}(P)$ (plotted vertically) against $\log_{10}(f)$?

1.3 Ready to study?

Study comment In order to study this module you will need to understand the following terms: *constant of proportionality*, *decimal places*, *dependent variable*, *dimensions*, *function*, *independent variable*, *index*, *inverse function*, *power*, *proportional*, *reciprocal* and *root*. You will need to be able to use SI units, perform simple algebraic and numerical calculations (including using a calculator), plot the graphs of simple functions, and determine the *gradient* of a straight line that may be specified graphically or algebraically. If you are uncertain about any of these terms you can review them by referring to the *Glossary*, which will indicate where in *FLAP* they are developed. The following *Ready to study questions* will allow you to establish whether or not you need to review some of the topics before embarking on this module.

You will find the answers to these questions in Section 5.

Question R1 Write the following expressions in their simplest form:

(a) $\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}$ (b) 5^0.

Question R2 If a is any positive number, what is the value of x in the equation: $(a^4)^5 \times (a^2)^3 = a^x$.

Question R3 Write the following expressions in their simplest form:

(a) $16^{-1/4}$, (b) $16^{3/4}$, (c) $4^{5/2}$, (d) $27^{-2/3}$, (e) $1/(3^{-2})$.

Question R4 If y is a function of x, given by $y = F(x)$, what is meant by saying that $G(x)$ is its inverse function? If $F(x) = x^3$, what is $G(x)$?

Question R5 Plot a graph of $y = x^2$ by evaluating the right-hand side of the equation at the points $x = 0, \pm1, \pm2, \pm3$. Use your graph to find solutions of the equation $x^2 = 2.72$.

Question R6 Which of the following expressions will give a straight line when y is plotted against x? For those that will give a straight line, state the value of its gradient. (All symbols except y and x represent non-zero constants.)

(a) $y = mx + c$, (b) $y = ax^2 + b$, (c) $y + x = k$, (d) $y/x = p$, (e) $y/x = qx + r$.

Question R7 What is the gradient of the straight line joining the points with Cartesian coordinates (1, 5) and (3, 13)?

2 Exponential functions

2.1 Exponential growth and decay

In physics, and elsewhere, we are often concerned with how a quantity changes with time. The following examples all have an important feature in common. As you read, think what that feature might be.

Suppose that you were to invest £100 with a bank at an interest rate of 5% per year. At the end of the first year your money would have earned £5 in interest, and your total investment would be worth £105. To find the value of your investment after a further year, you would add 5% of £105 (i.e. £5.25) to obtain a total of £110.25 — and so on. Year after year, your total investment would increase, and so would the annual interest, since it would grow in proportion to your total investment. Thus, on

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an annual basis, the rate of growth of your investment (i.e. the interest gained per year) is proportional to your total investment.

exponential change
exponential growth
exponential decay

When the electric charge Q stored in a capacitor is discharged through a resistor, the rate at which charge leaves the capacitor and flows through the resistor is described by the electric current I through the resistor. The size of this current is determined by the resistance R and the voltage V across the resistor: $I = V/R$. However, V itself depends on the capacitance C and the charge Q remaining in the capacitor: $V = Q/C$. It follows that $I = Q/(RC)$. So, at any moment, the rate at which charge is lost from the capacitor, I, is proportional to the charge, Q, remaining in the capacitor.

Charge, *current*, *voltage*, *resistance* and *capacitance* are all discussed in the *Glossary*, but if you are totally unfamiliar with them, just regard Q, I and V as related variables, and R and C as constants.

The 'activity' of a radioactive sample is a measure of the number of atomic nuclei in that sample that disintegrate per second. Since each disintegration effectively removes one unstable nucleus from the sample it is also a measure of the rate at which unstable nuclei are lost from the sample. Now any individual nucleus is equally likely to decay in each second of its lifetime, so the number of disintegrations occurring in a sample in one second will be proportional to the number of unstable nuclei in that sample. Thus, at any time, the rate of decrease in the number of the unstable nuclei in a sample is proportional to the number of unstable nuclei that remain.

Radioactivity is also discussed in the *Glossary*. Note that this example assumes that the decaying nuclei are all of the same type and that the nuclei produced by the decay are stable.

◆ What do the above examples have in common?

◇ Each describes a situation where the *rate of change* of some quantity at a given time is proportional to the value of that quantity at that time. □

All the changes discussed above are examples of **exponential changes**. Such change may cause a quantity to increase (e.g. to grow) or to decrease (e.g. to decay), and may be characterized in the following way:

> In an *exponential change* the rate of change of some quantity, y, at any time t, is directly proportional to the value of y itself at that time:
>
> $$\{\text{rate of change of } y(t)\} \propto y(t)$$
>
> i.e. $$\{\text{rate of change of } y(t)\} = ky(t) \qquad (1)$$
>
> where k is a constant of proportionality.
>
> If k is positive, y increases with time — this kind of change is called **exponential growth**.
>
> If k is negative, y decreases with time — this kind of change is called **exponential decay**.

The symbol $\propto$ is read as 'is directly proportional to'.

In our first example, the constant k was simply the interest rate, so, $k = 0.05$ year^{-1} (i.e. 5% per year). In the third example, the case of radioactive decay, the rate of change in the number of unstable nuclei was *negative* since the change *reduced* the number of such nuclei in the sample. (☞) In such cases the constant of proportionality is usually written as $-\lambda$, where λ is a positive quantity called the *decay constant* — for example, a certain isotope of polonium has a decay constant of $\lambda = 0.0133$ s^{-1}. Notice that the units of the decay constant reflect the units of the time interval used to define the rate.

If you are feeling confused about this negative sign, don't worry. Rates of change and their signs are discussed in detail in the next subsection.

Question T1 How many nuclear disintegrations per second would you expect from a sample containing 6.0×10^{18} polonium nuclei ($\lambda = 0.0133$ s^{-1})? □

You will find the answers to all *Text questions* in Section 5.

Question T2 In the second example above, what is the constant of proportionality relating the rate of discharge, I, of a capacitor to the charge Q remaining on the capacitor? □

There are many examples of exponential change in physics, some of which you will meet during this module. All exponential changes have an underlying mathematical similarity, and later in this section we will develop some powerful mathematical ideas and techniques relating to such changes. First, though, we need to have a more careful look at the idea of a *rate of change*.

2.2 Gradients and rates of change

Figure 1 illustrates how the volume, V, of water increases with time when a bath is filled. You can see from the graph that the water from the tap is running at a

Rates of change are also discussed (using different examples) in the *FLAP* module that introduces differentiation.

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constant rate, because the volume of water increases by equal amounts in equal time intervals, i.e. by 2 litres in each second — the rate of change of volume is therefore 2 litres s^{-1}.

◆ What is the gradient of the graph in Figure 1? What is the relationship between the gradient and the rate of change of volume?

◇ The gradient is 2 litres s^{-1}. The gradient of the graph is *equal* to the rate of change of volume. □

◆ What would the graph look like if the bath were *emptying* at a constant rate of 5 litres s^{-1}?

◇ The graph would still be a straight line, but sloping in the opposite direction, and steeper. Its gradient would therefore be negative, −5 litres s^{-1}. □

We can generalize the above discussion to *any* constant rate of change:

> For any quantity y that changes at a constant rate, the graph of y against time t is a straight line with a gradient equal to the rate of change of y.

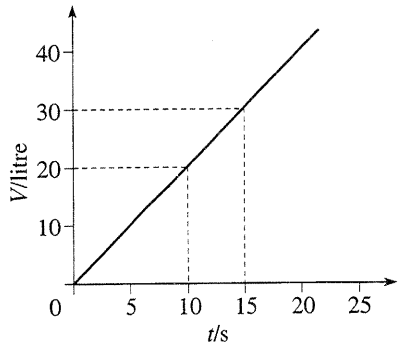


Figure 1 The volume of water in a bath that fills at a constant rate.

In practice, a bath does not continue to empty, unaided, at a constant rate. As the water level falls, the water pressure also falls, and so the rate at which water flows out of the bath decreases. Figure 2 shows how the volume of water in an emptying bath might change with time. The graph drops steeply at first, corresponding to a rapid flow, and gradually becomes shallower as the flow rate diminishes. As before, we can relate the rate of change of volume to the steepness of the graph, even though the steepness is changing from moment to moment, but how do we do this? What feature of a *curved* graph such as in Figure 2 will let us work out the rate of change of the plotted quantity at any time?

If conditions were such that the flow rate were proportional to the amount of water remaining, this would be another example of exponential change — but not otherwise.

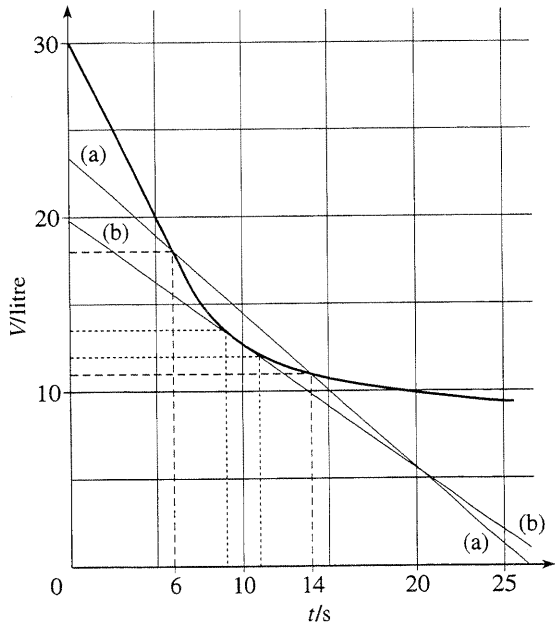


Figure 2 The volume of water in a bath emptying at a decreasing rate. The lines (a) and (b) are used to estimate the flow rate at t = 10 s.

We could get a rough value for the rate of change of volume at, say, t = 10 s by finding the change in the volume of water in the bath between t = 6 s and t = 14 s (about −7 litres) and then dividing that volume by the time interval of 8 s (14 s − 6 s) to get a rate of change of volume of about −0.9 litres s^{-1}. This is equivalent to finding the gradient of the straight line joining the points on the graph at t = 6 s and t = 14 s. Or we could choose a shorter time interval, say 2 s, between t = 9 s and t = 11 s. You can see from the lines (a) and (b) drawn on Figure 2 that a different choice of time interval gives a straight line with a different gradient, and hence a different rate of change of volume. The difference arises because the flow rate is changing during the time intervals and what we have calculated is an average rate of change of volume over each of the specified periods. But how can we improve on this to find the rate of change at a particular time?

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tangent

If we make the time interval *very small indeed*, we can hope that the flow rate hardly changes at all during that interval. Instead of a line joining two well-separated points on a curve, the situation is more like that shown in Figure 3; there will be single straight line that just touches the curve at $t = 10$ s, the steepness of which matches exactly that of the curve at that point. Such a line is called a **tangent** to the curve. Using this idea we can define the gradient of the curve, at any particular point, to be the gradient of the tangent to the curve at that particular point. This gradient tells us the *instantaneous* rate of change at our chosen value of t, rather than the average rate of change over an interval. So, if we have a change in volume ΔV in a time interval Δt, the quantity $\Delta V/\Delta t$ will be equal to the average rate of change of volume during the interval Δt. But if we make Δt and ΔV smaller and smaller we can reasonably expect that $\Delta V/\Delta t$ will provide an increasingly good estimate of the gradient of the tangent. Indeed, if Δt and ΔV are small enough, we can expect $\Delta V/\Delta t$ to represent the (instantaneous) rate of change of V.

ΔV (delta vee) means 'a small change in V'. It is *not* a factor Δ multiplied by the volume V, so the Δ *cannot be cancelled* in the expression $\Delta V/\Delta t$.

Exactly what is meant by 'small enough' is an important topic that occupies a good deal of the module that introduces differentiation.

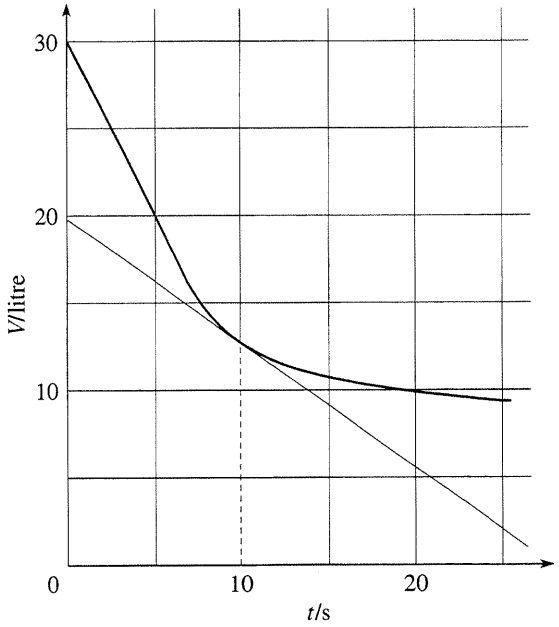


Figure 3 Finding the gradient at a point on a curve by drawing a tangent.

In the remainder of this module we will use the notation $\Delta V/\Delta t$ to represent the instantaneous rate of change of V with respect to t. In other words, no matter what value of t we are discussing, we will always assume that we can find suitable values Δt and ΔV to ensure that $\Delta V/\Delta t$ provides an accurate value for the gradient of the tangent at that value of t. From a strictly mathematical point of view this is not always justified, nor is it a particularly good use of notation, but we shall use it none the less.

Better ways of describing instantaneous rates of change are dealt with elsewhere in *FLAP*, particularly in the modules devoted to the subject of *differentiation*.

When trying to find an instantaneous rate of change from a graph you will probably have to make your best guess at an appropriate tangent, evaluate its gradient as accurately as possible, and accept that by working graphically you are limited to making *estimates* of rates of change. (Fortunately, there are algebraic techniques that enable us to work out rates of change accurately, but these too belong to the subject of *differentiation* and will not be developed in this module.)

Question T3 Find the gradient of the tangent at $t = 10$ s shown in Figure 3. Draw a tangent to the curve at $t = 5$ s and hence estimate $\Delta V/\Delta t$ when $t = 5$ s. □

The above discussion can be generalized to the rate of change of any quantity:

> The rate of change of any quantity y, at a particular time, can be represented by the quotient $\Delta y/\Delta t$, provided the changes Δy and Δt are sufficiently small. The value of such a rate of change is given by the gradient of the tangent to the graph of y against t at the time in question.

Furthermore, the idea of a gradient of a curved graph is not confined to graphs showing variation with time.

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◆ Figure 4 is a graph of the equation $y = x^2$. By drawing tangents to the curve, estimate $\Delta y/\Delta x$ when $x = 0$, $x = 2$, and $x = -1$.

◇ At $x = 0$, you can see from the symmetry of the curve that the gradient is 0. The gradients of the other two tangents are 4 and −2, respectively, though your own results might be slightly different from these values since they will depend on how accurately you draw the tangents. □

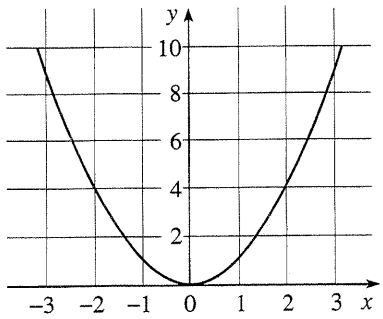


Figure 4 A graph of the equation $y = x^2$.

Before we end this subsection, we will have a brief look at some more rates of change and their physical interpretations. One example is shown in Figure 5a, where the graph shows how the position coordinate, x, of an object moving along a straight line, changes with time. This is called *linear motion*. Where the graph is steep, the position changes rapidly with time — i.e. the object moves quickly — and shallower parts of the graph correspond to the object moving more slowly. A negative gradient corresponds to the object moving in the reverse direction. We can therefore say that, at any particular time, the gradient of this position–time graph is equal to the instantaneous velocity v_x of the object. Figure 5b shows how the velocity, v_x, of the object represented in Figure 5a changes with time.

Linear motion is discussed in greater detail elsewhere in *FLAP*. See the *Glossary* for details.

◆ Suggest a physical interpretation of the gradient of Figure 5b.

◇ The gradient is $\Delta v_x/\Delta t$, the rate of change of velocity, which is equal to the acceleration. □

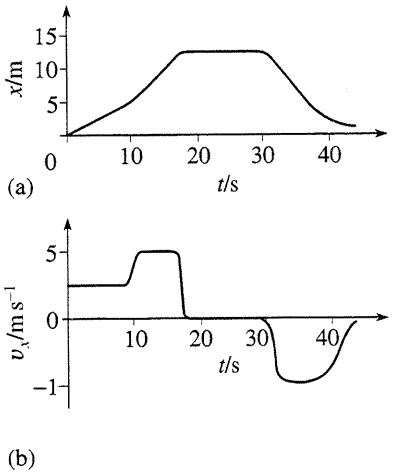


Figure 5 Graphs of (a) the position coordinate and (b) the instantaneous velocity of an object moving along a straight line.

Using the notation introduced above, we can rewrite the condition for exponential change (Equation 1) as:

In an exponential change, at any time $\Delta y/\Delta t = ky$ (2)

Figure 6 shows an example of a quantity that is changing exponentially: N, the number of unstable nuclei in a radioactive sample, decays exponentially with time t. The gradient of the curve at any particular time is equal to the number of disintegrations per second occurring at that time. By drawing tangents to the graph at various times and measuring their gradients it is easy to see that the gradient is indeed proportional to N, as required. When $N = 10 \times 10^{10}$, 6×10^{10} and 2×10^{10}, the measured gradients are -5×10^4 s^{-1}, -3×10^4 s^{-1} and -1×10^4 s^{-1}.

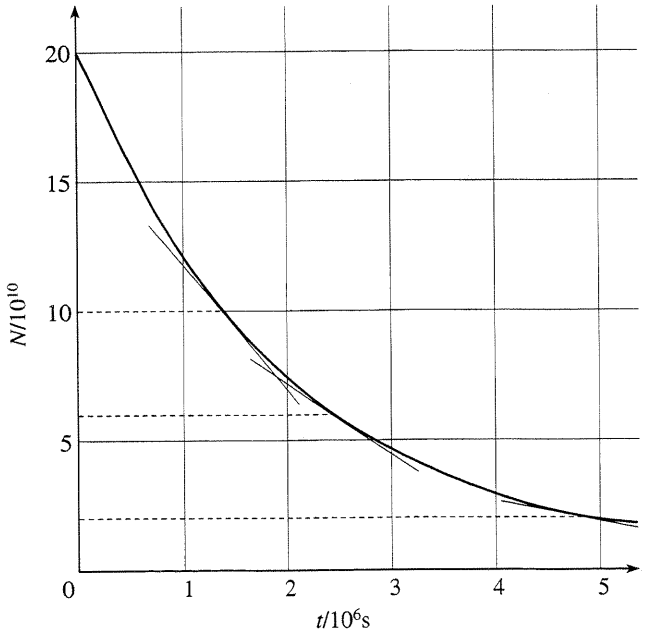


Figure 6 The number of unstable nuclei in a radioactive sample plotted against time, with tangents drawn at $N = 10 \times 10^{10}$, 6×10^{10} and 2×10^{10}. This is an example of exponential decay.

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Figure 7 shows how the charge Q (measured in coulombs) stored in a capacitor changes with time. By measuring the gradients of tangents at various times, we can find the rate of flow of charge, i.e. the current, at these particular times.

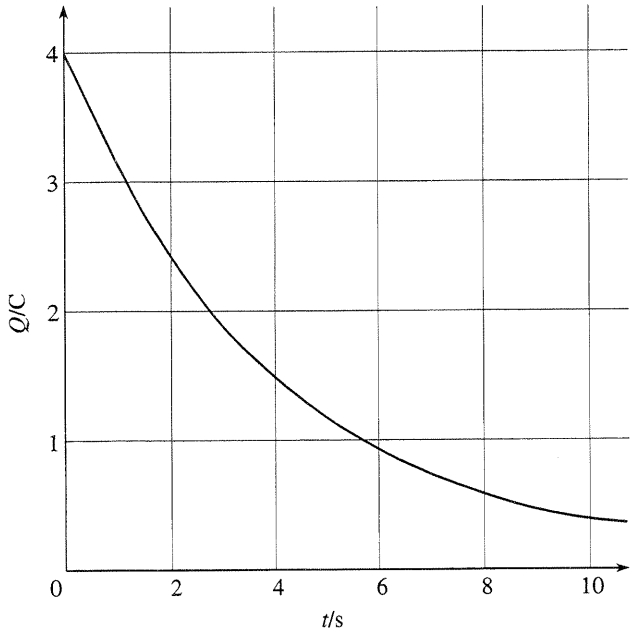


Figure 7 The charge remaining on a discharging capacitor. Another example of exponential decay.

Question T4 By drawing tangents to the curve, estimate the currents when $Q = 2$ C, 1 C and 0.5 C, and hence verify that Figure 7 shows exponential decay. □

Question T5 Figure 8 shows the graph of $V = 8/x^2$. Explain how you would show that this curve does not describe exponential decay. □

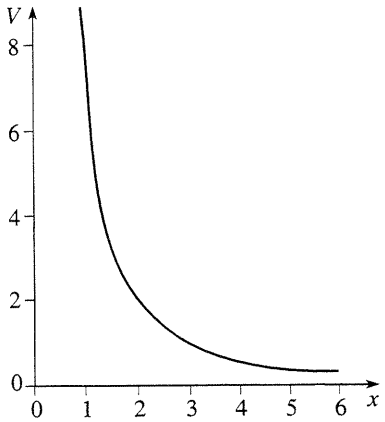


Figure 8 See Question T5.

2.3 Exponential functions and the number e

In this subsection, you will learn how to write *functions* representing the values of quantities that change exponentially. We will begin with exponential growth, since we can then deal entirely with positive quantities. We will look for a function $y(x)$ such that, at any value of x, the rate of change of y is equal to the value of y itself, i.e. we will look for a function $y(x)$ such that $\Delta y/\Delta x = x$. For the sake of simplicity we will treat x as a purely numerical variable.

Remember that in exponential growth the constant k is positive, whereas in exponential decay k is negative.

In Subsection 2.1, we saw that when there is a constant annual interest rate the value of an investment grows exponentially. With an interest rate of 5% per year, the value after one year is found by multiplying the initial sum by 1.05 and, after two years, by multiplying again by 1.05, i.e. in two years the value of the initial investment increases by a factor of $(1.05)^2$. If the investment is left for n years at the same interest rate, its initial value will be multiplied by a factor of $(1.05)^n$.

The above example suggests that we should look at functions of the form $y(x) = y_0 a^x$, where y_0 is the initial value of y and a is some (positive) constant number. It may therefore be helpful at this point to have a brief reminder of the properties of such functions. For any numbers a, x and y:

$$a^x a^y = a^{x+y} \tag{3}$$

$$(a^x)^y = a^{xy} \tag{4}$$

$$a^{-x} = \frac{1}{a^x} \tag{5}$$

$$a^{x/y} = (a^{1/y})^x \tag{6}$$

The relationships expressed in Equations 3 to 6 are derived and discussed elsewhere in *FLAP*. See the entry on *powers* in the *Glossary* for details.

Also, recall that $a^0 = 1$ and that by $a^{1/k}$ (for k a positive integer) we mean the kth root of a, i.e. a solution of the equation $x^k = a$.

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Now let us look at a particular function: $y(x) = 2^x$. Figure 9 shows the graph of this function, and Table 1 shows its value for various values of x.

Table 1 Values of the function $y\ (x) = 2^x$.

x	2^x	x	2^x
−3.5	0.09	0	1.00
−3.0	0.13	0.5	1.41
−2.5	0.18	1.0	2.00
−2.0	0.25	1.5	2.83
−1.5	0.35	2.0	4.00
−1.0	0.50	2.5	5.66
−0.5	0.71	3.0	8.00

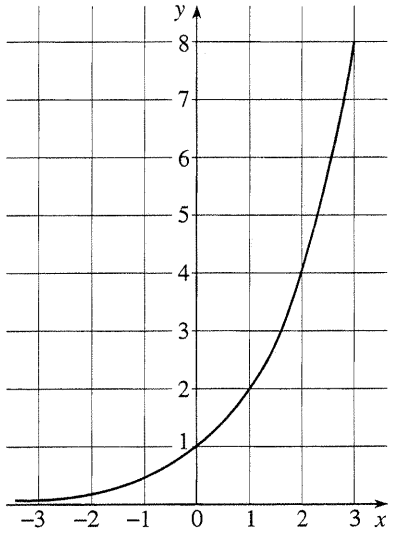


Figure 9 The graph of $y(x) = 2^x$.

◆ Find the gradients of tangents to the graph of $y = 2^x$ at $x = -1$, 0, 1 and 2. Is this function growing exponentially?

◇ At the given values of x the gradients are approximately 0.7, 1.4 and 2.8, respectively. From Table 1 the corresponding values of y are 1.0, 2.0 and 4.0. So, the gradient apparently increases in proportion to the value of the function. This indicates that $y(x) = 2^x$ increases exponentially with increasing x. □

In fact the gradient is consistent with $\Delta y/\Delta x \approx 0.7y$, not with $\Delta y/\Delta x = y$. If you were to plot the graph of $y(x) = 3^x$ you would find that it too grows exponentially, with a gradient approximately equal to 1.1×3^x. This suggests that there may be some number a between 2 and 3 such that if $y(x) = a^x$, then the gradient would be exactly equal to the value of y.

It turns out that there is indeed such a number. To three decimal places its value is 2.718, but like π and $\sqrt{2}$, it is an *irrational* number that cannot be accurately represented by any decimal with a finite number of decimal places. For this reason it is conventional to represent its accurate value by the letter e and simply substitute an appropriate numerical value whenever necessary. e is one of the most important numbers in physics and in mathematics.

e = 2.718 281 828 459 05 ... (to 14 decimal places)

If $y(x) = e^x$ (7)

then the rate of change $\Delta y/\Delta x = y(x)$

◆ Figure 10 is a graph of $y(x) = e^x$. Draw tangents at a few different points on the graph and measure their gradients. Confirm that at each point you examine the gradient is equal to the value of y.

◇ You should have found that each gradient is close to the value of $y(x)$, thus confirming that the function satisfies $\Delta y/\Delta x = y$. □

The function e^x is known as the **exponential function**, and to emphasize that it is a *function* of x it is often written as exp (x). So far, we have used it to describe how something varies with x, where x can be any variable without dimensions (since powers are always dimensionless numbers).

To evaluate exponential functions on a calculator, use the e^x key (possibly labelled exp (x)) or its equivalent. If your calculator does not have such a key, you may be able to calculate e^x by using the y^x key with $y = 2.718$, but you are probably best advised to buy a new calculator.

Question T6 Evaluate e^2, e^3, $e^{1.43}$, e^{-1} and e^0. □

e
exponential function

An *irrational number* is one that cannot be written in the form p/q where p and q are whole numbers (i.e. integers).

In Subsection 2.5, you will see how the value of e can be calculated to any desired degree of precision.

e^x can be written as exp (x).

In this module we include parentheses in terms such as exp (x), but it is conventional to drop parentheses in cases such as these, and to only include them to avoid ambiguity.

BEWARE! A calculator key labelled EE will give you 10^x, not e^x. Even more confusingly, on some calculators you will find a button marked EXP for 10^x.

In Section 3 you will see that there is another way to evaluate exponential functions on a calculator.

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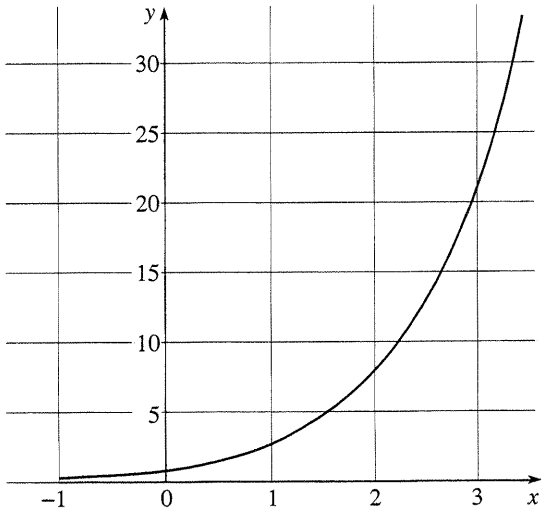


Figure 10 The graph of the function $y(x) = e^x$.

2.4 Exponential functions and exponential change

So far, you have seen that the function $y(x) = \exp(x)$ describes exponential changes in which the rate of change is equal to $y(x)$ at any given value of x. In this subsection, we will see how exponential functions can be used to describe exponential changes that correspond to any value of k in $\Delta y/\Delta x = ky$.

◆ Table 2 gives some values for the function $y(x) = \exp(3x)$. Figure 11 shows the graph of this function for the range $x = 0$ to $x = 0.6$. Draw tangents at the points $y = 2$ and $y = 3$ and measure their gradients. Suggest a relationship between the values of y and the gradient.

◇ At any value of y, the gradient is $3y$. Your measured values should have been close enough to the true values (6 and 9) to suggest this result. □

Table 2 A table of values (to three decimal places) for the function $y(x) = \exp(3x)$.

x	$\exp(3x)$
−1.0	0.050
−0.5	0.223
0	1.000
0.5	4.482
1.0	20.086
1.5	90.017
2.0	403.429

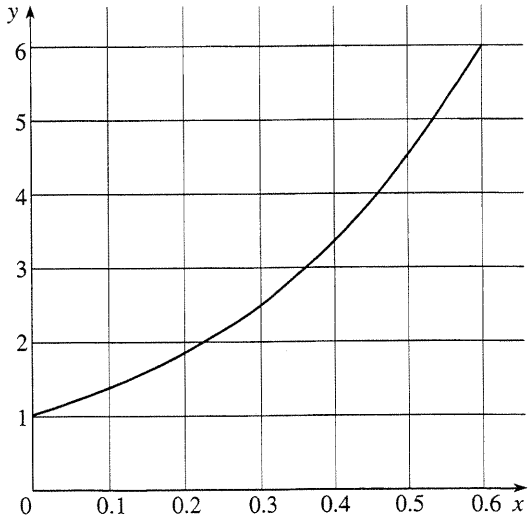


Figure 11 The graph of $y(x) = \exp(3x)$.

You have just illustrated an important general rule about exponential functions:

> For any function $y(x) = \exp(kx)$, the gradient at any value of x is $ky(x)$, and so the rate of change $\Delta y/\Delta x = ky(x)$.

In other words, we seem to have found the function that satisfies the general condition for exponential change. So far, though, we have looked only at examples where k is positive (i.e. exponential growth), whereas we also need to be able to deal with exponential decay, where k is negative.

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◆ Figure 12 shows a graph of the function $y(x) = e^{-x}$ (i.e. $k = -1$). What is the sign of the gradient at any point on the curve? What is the relationship between the gradient and the value of y? (Draw tangents and measure their gradients if you have to, but you can probably guess the answer.)

◇ The gradients are all negative. At any value of x the gradient is $-y(x)$. □

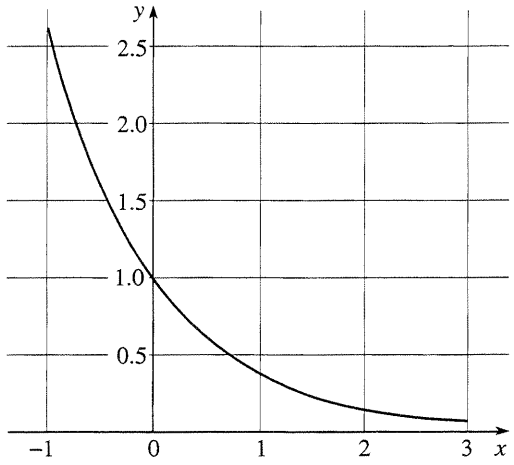


Figure 12 The graph of the function $y(x) = \exp(-x)$.

The above example illustrates that the exponential function $y(x) = \exp(kx)$, with a suitable value of k, can describe both exponential growth and exponential decay. So we now have an almost complete description of any exponential change. Why 'almost'? So far, we have neglected the initial value of y. If $y = \exp(kx)$, then when $x = 0$, $y = 1$. Clearly, this is not necessarily true in all physical situations. Also, since the start of Subsection 2.3 we have been treating x and y as a purely numerical variables, without any associated units. In practice we are certain to be interested in situations where y and x are physical quantities that involve units of measurement.

Dealing with these remaining conditions is actually very straightforward. All we have to do to represent a general exponential change is to use a function of the form $y_0 \exp(kx)$. Thus:

For any exponential change, i.e. any change in which $\Delta y/\Delta x = ky$

$$y(x) = y_0 \exp(kx) \qquad (8)$$

where y_0 is the value of y when $x = 0$.

As far as units are concerned, y will have the same units as y_0, and the product kt must be a pure number (i.e. dimensionless), so if x is a time in seconds, say, then the units of k should be seconds^{-1}. (☞)

You may well be asked to deal with cases in which this last condition is not satisfied. For instance, x may be quoted in seconds, but k may be given in hours^{-1}. If so, you should convert one or other of the quantities, to ensure that kx is a pure number, before evaluating e^{kx}.

Figure 13 shows the graphs of the exponential function $y = y_0 e^{kx}$ for various values of y_0 and k. If you measured the gradients of tangents to any of these graphs you would find that in each case they satisfy the equation $\Delta y/\Delta x = ky$ at every value of x. This illustrates the general rule that for exponential functions *the initial value of y does not alter the relationship between the value of y and the gradient.* So, we now have a complete 'recipe' for describing exponential change.

Question T7 When a capacitor discharges, $\Delta Q/\Delta t = -Q/(RC)$. If the initial charge is Q_0, write an equation that describes how Q changes with time (i.e. write a definition of Q as a function of time). □

Question T8 Radioactive decay may be described by the equation $N = N_0 e^{kt}$. What are the values of N_0 and k for the decay shown in Figure 6? □

We will now consider some specific examples of exponential change.

Half-lives and decay absorption coefficients

Apart from the proportionality between the function and the value of its gradient, $y(t) = y_0 e^{kt}$ has another interesting property. To see this, consider the effect of increasing t in a series of equal steps, say from 0 to $1/k$, to $2/k$, to $3/k$, and so on. What are the corresponding values of $y(t)$? They are y_0, $y_0 e$, $y_0 e^2$, $y_0 e^3$, and so on. Clearly, equal additions to t cause $y(t)$ to change by equal multiplicative factors.

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This is true whatever the size of the additions to t may be (though different sized increments naturally correspond to different multiplicative factors), and it applies to exponential decay as well as exponential growth. In fact, the best known example of this is probably the use of **half-life** to characterize the decay of radioactive nuclei. Such decays are described by an exponential function of the form $N(t) = N_0 e^{-\lambda t}$, so in any time interval of length $1/\lambda$ the number of nuclei will decrease by a factor of e^{-1}. Similarly, in any interval of length $0.693/\lambda$ the number of nuclei is reduced by a factor of 0.5. That's why this is called the half-life. (☞)

half-life

$0.693/\lambda$ is accurate to three decimal places, but the precise relationship between the half-life and the characteristic decay constant λ involves the *logarithmic function* introduced in the next section. This relationship will be discussed in Question T17.

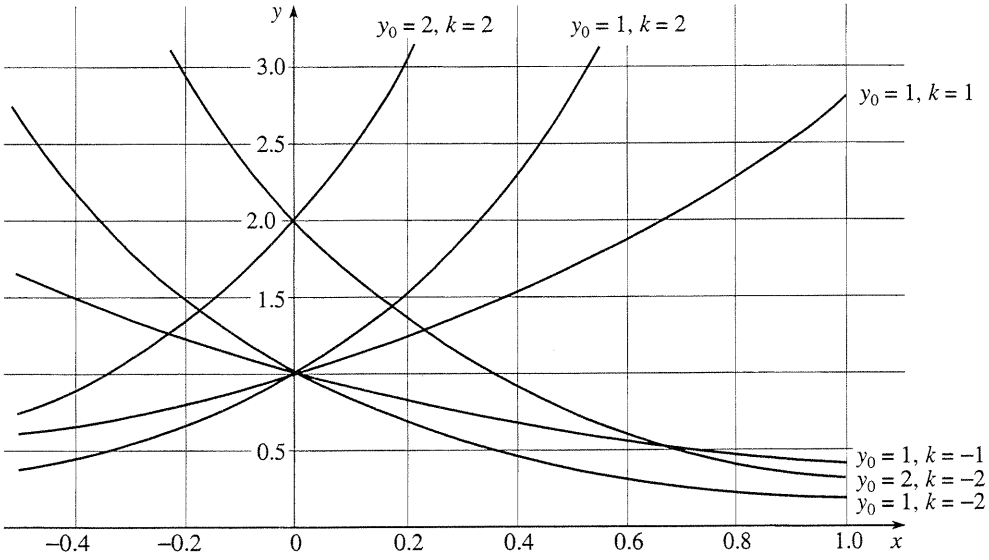


Figure 13 Graphs of the function $y(x) = y_0 e^{kx}$ for various values of y_0 and k.

So far we have been concentrating on changes with time, but we have already noted that an exponential function is not limited to such changes. For example, when a parallel beam of electromagnetic radiation passes through matter, its intensity I (☞) is related to the distance x that it has travelled through the medium by an exponential function: $I = I_0 \exp(-\mu x)$ where μ is called the *absorption coefficient*. It also depends on the nature of the radiation and the nature of the matter. For every additional distance $1/\mu$ that the beam travels through the matter, its intensity is reduced by a factor of e^{-1}.

Intensity, I, is the rate at which energy is transferred across unit area. See the *Glossary* for details.

◆ What are the dimensions of μ?

◇ Since μx must be dimensionless μ must be reciprocal length, $[L^{-1}]$. □

The function a^t

Throughout the last two subsections, we have been mainly concerned with functions of the form $y(x) = y_0 e^{kx}$. But we have also seen that functions such as 2^x and 3^x also satisfy the requirement that their gradients are proportional to their values at any given value of x. Clearly then, the exponential function $\exp(x)$, or e^x, is just one special example of the kind of function that can describe exponential change. More generally, any function of the type $y(x) = y_0 a^{kx}$ (where a is any positive number) will describe exponential change, though it will *not* satisfy the equation $\Delta y/\Delta x = ky$. For instance, in Subsection 2.3, you saw that the gradient of the graph of $y(x) = 2^x$ is approximately $\Delta y/\Delta x \approx 0.7y$ while that of $y(x) = 3^x$ is approximately $\Delta y/\Delta x \approx 1.1y$, so in both of these cases, where $k = 1$, it is certainly not the case that $\Delta y/\Delta x = ky$ even though $\Delta y/\Delta x$ is proportional to y. What really distinguishes the function $y(x) = y_0 e^{kx}$ is the fact that *for it alone* we can assert that $\Delta y/\Delta x = ky$ for all values of k.

Although the functions $y(x) = y_0 e^{kx}$ and $y(x) = y_0 a^{kx}$ are different (provided $a \neq e$), there is a simple relationship between them. Because it is always possible to find a number c such that $a = e^c$, for any positive value of a, it is always possible to write

$$y_0 a^{kx} = y_0 (e^c)^{kx} = y_0 e^{ckx} \quad (a > 0)$$

Thus, any function that describes exponential change can always be rewritten in terms of the exponential function, all we have to do is find the value of c that satisfies the equation $a = e^c$. We will return to this in Subsection 3.2, where you will see how it is done.

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A note on terminology In an expression such as q^p (where p and q are either constants or variables), the power p is sometimes called the **exponent** of q. Correspondingly, some authors use the term 'exponential function' to mean *all* functions of the type $y(x) = a^x$. Those authors still use exp (x) to represent e^x, but they sometimes call it the **natural exponential function** (since it arises from descriptions of naturally-occurring processes) in order to distinguish it from other functions of the type a^x. However, in *FLAP* the term exponential function is generally used to mean a function of the form $y(x) = e^x$ or $y(x) = e^{kx}$. Variables related by functions of the form $y(x) = y_0e^{kx}$ are generally said to satisfy **exponential laws**. Such laws arise in many areas of physics.

exponent
natural exponential function
exponential law

p is also known as the *index*, or the *power*, of q.

2.5 Evaluating the number e

In Subsection 2.3, the value of e was produced more or less out of a hat, and was then shown to have the necessary properties to describe exponential change. Now we will show how the value of e can be calculated from first principles.

We start from the requirement that we want a number e such that if $y(t) = y_0e^{kt}$, then $\Delta y/\Delta t = ky$. In order to avoid having to deal with negative numbers, we will consider an example of continuous exponential growth. Our original example of the growth of an investment is now not a very good one, since the interest is added only once a year and so the value goes up in steps rather than continuously. A better example would be the growth of a large population of organisms (aphids, perhaps, or bacteria) where the population changes over even a very short time interval.

Suppose the number of organisms at time t is $N(t)$, and that at any time t the instantaneous rate of change is $\Delta N/\Delta t = kN(t)$. Even if we do not have an expression for $N(t)$ in terms of t, we can still use the given information to draw an *approximate* graph of N against t. To do this we note that the increase in the population over a small time interval Δt will be approximately $\Delta N = kN(t)\Delta t$. (This is only an approximation because the equation $\Delta N/\Delta t = kN(t)$ will not be completely accurate for any finite value of Δt, no matter how small.) So, if N_0 is the number of organisms at the *start* of a time interval Δt, the approximate number at the end of that interval will be $N_0 + \Delta N = N_0 + kN_0\Delta t$. This process can be repeated for the next interval Δt, by taking $N_1 = N_0 + kN_0\Delta t$ as the new starting value and $\Delta N = kN_1\Delta t$ as the increment over that second interval. This can be continued until we have built-up a complete (though approximate) graph of $N(t)$ over any desired period of time. Figure 14a shows such a graph for $k = 0.1\ s^{-1}$ and $N_0 = 10^5$, drawn using a time interval of $\Delta t = 5$ s. If a time interval of $\Delta t = 1$s were used (still with $k = 0.1\ s^{-1}$), then the graph would not only be smoother but it would also climb more steeply, as shown in Figure 14b. The increased steepness arises because, instead of using the same rate of change for a whole 5 s, based on the value of N at the start of that interval, we calculate a new rate of change after only 1 s, based on a slightly larger N. Table 3 shows the values used to plot Figure 14.

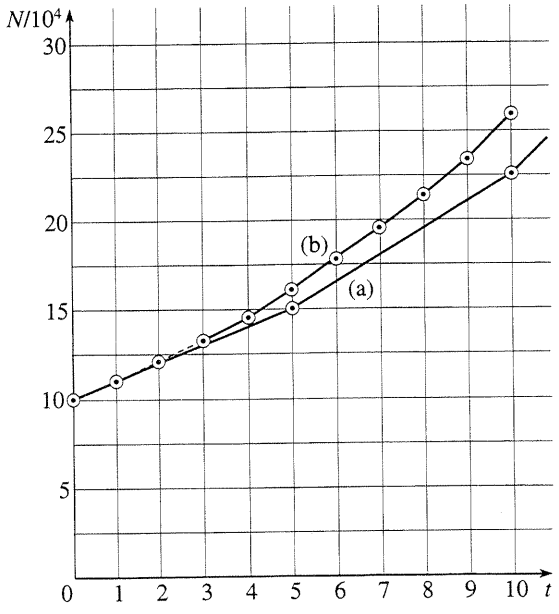


Figure 14 An approximate graph of N against t where $N_0 = 10^5$, $k = 0.1\ s^{-1}$ and (a) $\Delta t = 5$ s, (b) $\Delta t = 1$ s.

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Table 3 Values of N calculated using (a) $\Delta t = 5$ s, and (b) $\Delta t = 1$ s. (Remember that $N/10^4$ in the column heading means that each entry in the column has been divided by 10^4.)

t/s	(a) $N/10^4$ when $\Delta t = 5$ s	(b) $N/10^4$ when $\Delta t = 1$ s
0	10.000	10.000
1		11.000
2		12.100
3		13.310
4		14.641
5	15.000	16.105
6		17.716
7		19.487
8		21.436
9		23.579
10	22.500	25.937

◆ Which graph in Figure 14 is the better approximation to the real situation? How could the graphs be made even more realistic?

◇ Figure 14b is a better approximation because it follows the changes in N and $\Delta N/\Delta t$ more closely. The graphs can be made even more realistic by choosing a smaller time interval. □

Let us now see what happens as we make Δt smaller and smaller. Figure 15 illustrates the procedure. N_0 is the initial number of organisms and t is some arbitrary period of time which can be subdivided into intervals. If we divide t into 2 equal intervals of length $\Delta t = t/2$, and if N_1 is the approximate number after the first interval of length $t/2$, then

$$N_1 = N_0 + \frac{kN_0t}{2} = N_0\left(1 + \frac{kt}{2}\right) \tag{9}$$

and after the second interval of length $t/2$

$$N_2 = N_1 + \frac{kN_1t}{2} = N_1\left(1 + \frac{kt}{2}\right) \tag{10}$$

Combining Equations 9 and 10, we find

$$N_2 = N_0\left(1 + \frac{kt}{2}\right)^2 \tag{11}$$

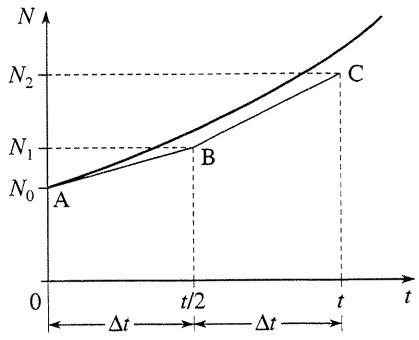


Figure 15 Estimating $N(t)$ by dividing a the time t into small intervals. The curve represents the actual variation of N with t.

Now, if we had chosen to use 10 smaller intervals, each of length $t/10$, we would have obtained a somewhat different, and more accurate, estimate of the total number of organisms after time t:

$$N_{10} = N_0\left(1 + \frac{kt}{10}\right)^{10}$$

Similarly, if we had used n equal intervals, then we would have found

$$N_n = N_0\left(1 + \frac{kt}{n}\right)^n \tag{12}$$

If we now let the value of n become larger and larger we can expect the value of N_n to get closer and closer to the true final value (which we know to be N_0e^{kt}). Thus, as n becomes larger the following approximation becomes increasingly accurate:

$$N_0e^{kt} \approx N_0\left(1 + \frac{kt}{n}\right)^n$$

For reasons that will soon become apparent, it is useful to eliminate n from this relationship in favour of a new variable m defined by $m = n/kt$. In making this substitution it is important to remember that the final power of n must be replaced

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by mkt, and the statement that the approximation becomes increasingly accurate as n becomes larger should also be restated in terms of m. Thus, as m becomes larger the following approximation becomes increasingly accurate:

$$N_0 e^{kt} \approx N_0\left(1+\frac{1}{m}\right)^{mkt} = N_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{kt} \qquad (13)$$

It follows from Equation 13 that $e \approx (1 + 1/m)^m$, and that this approximation becomes increasingly accurate as m becomes larger and larger. This statement can be made more concise by using the mathematical idea of a **limit**. We say that e is the *limit* of $(1 + 1/m)^m$ as m tends to infinity. This is conventionally written as:

limit

$$e = \lim_{m\to\infty}(1+1/m)^m \qquad (14)$$

The symbol ∞ represents infinity.

Question T9 Given that $a = (1 + 1/m)^m$, complete Table 4 (using a calculator) and thus confirm that $(1 + 1/m)^m$ provides an increasingly good approximation to e as m becomes larger and larger. □

Table 4 See Question T9.

m	a
2	2.2500
5	2.4883
10	2.5937
10^2	2.7048
10^3	
10^4	
10^5	

3 Logarithmic functions

In Section 2, you saw how exponential changes can be described by a function of the general form $y(t) = y_0 \exp(kt)$. Given values for y_0 and k, this function provides a unique value of y for any given value of t. In this section our main aim is to investigate the *inverse function*, that tells us the value of t corresponding to any given value of y. However, we begin our investigation by examining a slightly different question: given that $x = 10^y$, what is the value of y that corresponds to a given value of x? In other words, given that $x = 10^y$, we want to know the inverse function that will enable us to write y as a function of x.

3.1 Logarithms to base 10: the inverse of 10^x

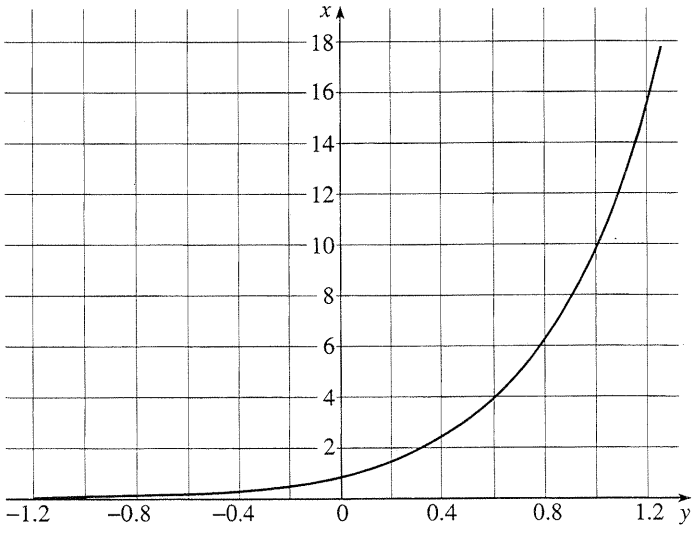
For some values of x, we can find y such that $x = 10^y$, without really thinking about inverse functions at all. For example, if $x = 100$ then, since $100 = 10 \times 10 = 10^2$, y must be equal to 2.

◆ If $x = 10^y$, what is y when, (a) $x = 10\,000$, and (b) $x = 0.1$?

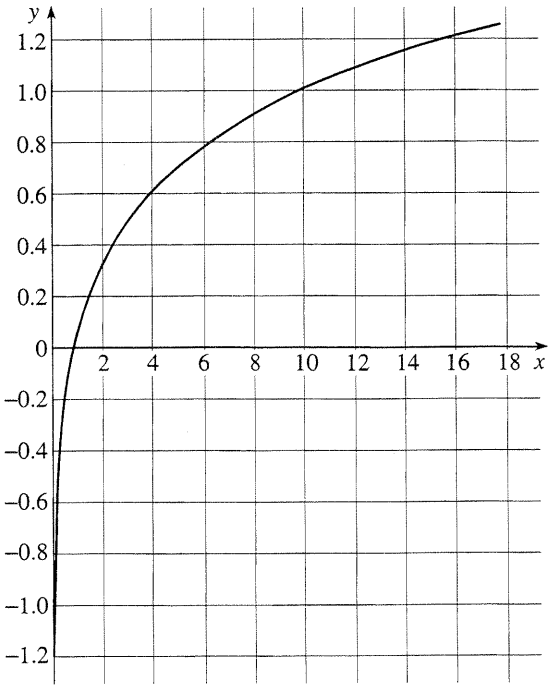
◇ $10\,000 = 10^4$, so $y = 4$; (b) $0.1 = 10^{-1}$, so $y = -1$. □

Figure 16a is a graph of the equation $x = 10^y$, plotted with the x-axis vertical and the y-axis horizontal. This may look odd, but when plotting graphs it is conventional to

Figure 16 (a) A graph of the function $x = 10^y$ with the x-axis vertical and the y-axis horizontal. (b) A graph of the inverse function, $y = \log_{10}(x)$, plotted on axes with the conventional orientation.



(a)



(b)

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logarithm to base 10
common logarithm
logarithmic function

plot values of the independent variable along the horizontal axis and values of the dependent variable along the vertical axis, and that is exactly what we have done. It is the choice of names for the variables that is unconventional in this case, not the graph plotting. In any event, it is clear from Figure 16a that each value of x corresponds to a different value of y, and that means it will be possible to define an inverse function to $x = 10^y$. The graph of this inverse function (with conventional x- and y-axes orientation) is shown in Figure 16b. It was obtained by re-plotting the data from Figure 16a so as to show y as a function of x. (This process is equivalent to reflecting the curve in Figure 16a in an imaginary mirror placed along the line $y = x$.) This shows that we can, in principle, find y for *any positive* value of x.

The function of x shown in Figure 16b is usually denoted by $y = \log_{10}(x)$ and its value for a given value of x is called the **logarithm to base 10** of x, or the **common logarithm** of x. This is an example of a **logarithmic function** —you will meet others in Subsection 3.2. As you can see from Figure 16b, it is only defined for positive values of x.

> The logarithm to base 10 of x is the number y which satisfies the equation $x = 10^y$, i.e.
>
> $$y = \log_{10}(x) \qquad (15)$$
>
> In other words, $\log_{10}(x)$ is the power to which we must raise 10 to obtain x.

We have written x in brackets in $\log_{10}(x)$, to emphasize that we are dealing with a *function*, but the brackets are often omitted in practice. Note, too, that $\log_{10}(x)$ is sometimes written as $\log(x)$.

From our definition we see that:

$\log_{10}(10) = 1$ since $10^1 = 10$

$\log_{10}(1) = 0$ since $10^0 = 1$

$\log_{10}\left(\frac{1}{10}\right) = -1$ since $10^{-1} = \frac{1}{10}$

Question T10 Without using your calculator, work out the values of:
(a) $\log_{10}(100)$, (b) $\log_{10}(1000)$, (c) $\log_{10}(0.1)$, (d) $\log_{10}(0.001)$, (e) $\log_{10}(10^{1/2})$, (f) $\log_{10}(10)$, (g) $\log_{10}(1)$, and (h) $\log_{10}(10^{1.52})$. □

◆ Use the x^y key on your calculator to find $10^{0.397\,94}$ and $10^{-2.301\,03}$, and hence find values for $\log_{10}(2.5)$ and $\log_{10}(0.005)$ to five decimal places.

◇ $\log_{10}(2.5) = 0.397\,94$, and $\log_{10}(0.005) = -2.301\,03$. □

The values of $\log_{10}(x)$ are harder to calculate directly when x is not an obvious power of 10. However, they can be read from graphs such as that in Figure 16b, or found using the 'log' key on a calculator. For example, to find $\log_{10}(3.7)$, key in 3.7 then press the 'log' function key —you should obtain the answer 0.568 2017.

◆ Use your calculator to find $\log_{10}(100)$, $\log_{10}(10)$, $\log_{10}(1)$, $\log_{10}(0.001)$, $\log_{10}(2)$ and $\log_{10}(3.16)$.

◇ $\log_{10}(2) = 0.301$, and you can check your other answers with the answer to Question T10. Notice that, to two decimal places, $3.16 = 10^{1/2}$. □

Question T11 Use the log function key on your calculator to find x when (a) $10^x = 6.8$, (b) $10^x = 537$, (c) $10^x = 0.34$. (Give your answers to four significant figures.) □

Question T12 Use your calculator to find (a) $\log_{10}(4.725)$, (b) $\log_{10}(47.25)$ and (c) $\log_{10}(472.5)$. (d) Without further use of the calculator, find $\log_{10}(4725)$ and $\log_{10}(4.725 \times 10^7)$. □

We have already established that taking the logarithm to base 10 'undoes' the operation of raising 10 to a power (see part (e) of Question T10, for example). Likewise, raising 10 to a power 'undoes' the operation of finding the logarithm to base 10. This can be expressed another way. If we eliminate y from Equation 15 we find

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$$10^{\log_{10}(x)} = x$$

therefore $\log_{10}(10^x) = x$ (16)

antilog
antilogarithmic function
base
logarithm to base *a*
logarithm to base e
natural logarithm

The function 10^x is sometimes called the **antilog** or **antilogarithmic function**. Antilogs can be found on a calculator using the 'inverse' and 'log' function keys.

Remember that 10^x and antilog (x) are identical.

◆ Use the function keys on your calculator to find antilog (x) for $x = 2$, $x = -1$, $x = 0.397\ 94$ and $x = -2.301\ 03$ — key in the number, then press 'inv' and 'log'.

◇ antilog (2) = 100, antilog (–1) = 0.1, antilog (0.397 94) = 2.5, antilog (–2.301 03) = 0.005. Notice that you have seen some of these results before, in Question T10 and the subsequent seeded question. □

3.2 Logarithms to base e and other bases

In Subsection 3.1, we looked at logarithms based on powers of 10. It is possible to use numbers other than 10 as the **base** for logarithms — in fact it is possible to have logarithms to *any* base a that is greater than zero:

The **logarithm to base *a*** of x is the number y such that $x = a^y$.

In other words, the logarithm to base a of x is the power to which we must raise a to obtain x, i.e. if $x = a^y$ then

$$y = \log_a(x) \quad (17)$$

Just as the functions 10^x and $\log_{10}(x)$ are the inverse of each other, so the functions a^x and $\log_a(x)$ are the inverse of each other. This can be expressed by eliminating y from Equation 17 to obtain $x = a^{\log_a(x)}$, or alternatively by eliminating x to obtain $y = \log_a(a^y)$. Replacing y by x throughout the second of these results (which we are free to do since we can always rename a variable) we see that for any base a:

if $a^{\log_a(x)} = x$

then $\log_a(a^x) = x$ (18)

◆ Given that $N/N_0 = 2^{kT}$, find an expression for T.
(*Hint*: Start by taking appropriate logs of both sides of the given equation.) (☞)

The process of 'taking logs of both sides' is the standard method of extracting a variable from an exponent. Similarly, you can often extract a variable from the argument of a logarithm by 'exponentiating both sides' of an equation.

◇ Taking $\log_2$ of both sides, we find $\log_2(N/N_0) = \log_2(2^{kT})$

thus, from Equation 18, $\log_2(N/N_0) = kT$

so, $T = (1/k)\log_2(N/N_0)$ □

The most widely-used logarithms and logarithmic functions are those based on powers of 10 or on powers of the number e.

The **logarithm to base e** of x is the number y such that $x = e^y$, i.e.

if $x = e^y$

then $y = \log_e(x)$ (19)

and hence

$$\exp[\log_e(x)] = x$$

and $\log_e[\exp(x)] = x$ (20)

Since the number e arises from the description of naturally-occurring processes (radioactive decay), the logarithmic function based on powers of e is often known as the **natural logarithm** and $\log_e(x)$ is sometimes denoted by $\ln(x)$. Note that $\log_e(x)$ can be found on a calculator, in the same way as finding $\log_{10}(x)$, the key is usually marked $\log_e(x)$ or $\ln(x)$. (You may have to press 'inv' followed by 'e^x' if your calculator doesn't have a '$\log_e$' or 'ln' key.)

Natural logarithms are also known as Naperian logarithms or hyperbolic logarithms.

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◆ Given that e = 2.718 (to three decimal places), use the y^x key on a calculator to find $e^{1.5041}$, and hence write down an approximate value for $\log_e(4.5)$.

◇ $e^{1.5041} = 4.500$ (to three decimal places), so $\log_e(4.5) \approx 1.5041$. You can check the answer by using the $\log_e(x)$ key on a calculator. □

Question T13 Without using your calculator, what are the values of $\log_e(1)$ and $\log_e(e)$? □

Question T14 Using your answers to Question T13, and the value of $\log_e(10)$ found using your calculator, sketch an approximate graph of the function $\log_e(x)$ on the same axes as a graph (such as that in Figure 16b) of $\log_{10}(x)$. □

The relationship between logs of different bases is discussed further in Subsection 3.3 of this module.

Natural logarithms can, as you have just seen, be handled on a calculator in a similar way to common logarithms. Their inverse function, too, can be dealt with similarly.

◆ Suggest how to find $\exp(x)$ on a calculator using the 'inv' (i.e. inverse) and 'log' keys. Use this method to find $e^{1.5041}$.

◇ Key in 1.5041, then press the inverse and $\log_e$ function keys. You should obtain a value very close to 4.5. □

The term 'antilog' is normally used *only* to denote the inverse of the logarithm in base 10.

We are now able to solve the problem posed at the start of Section 3 — namely, to write t as a function of y given that $y(t) = y_0 \exp(kt)$. If we divide both sides of this equation by y_0 we obtain $y/y_0 = \exp(kt) = e^{kt}$. We can then use Equation 21 (or take $\log_e$ of both sides) to deduce that:

if	$y = y_0 \exp(kt)$	
then	$\log_e(y/y_0) = kt$	(21)
so that	$t = (1/k)\log_e(y/y_0)$	

When dealing with exponential decay, the constant k will be negative, and care will need to be taken when dealing with signs. For example, when a capacitor discharges through a resistor, the charge Q on the capacitor is given as a function of time by $Q = Q_0 \exp(-t/RC)$. To find the time at which Q reaches a certain value, divide both sides by Q_0 to obtain $Q/Q_0 = \exp(-t/RC)$, and then take $\log_e$ of both sides to see that $\log_e(Q/Q_0) = -t/RC$. So, $t = -RC \log_e(Q/Q_0)$.

We can also now solve the problem encountered at the end of Subsection 2.4, where we wanted to find a number c such that $a^{kt} = \exp(ckt)$. In other words, since $e^{ckt} = (e^c)^{kt}$, we wanted to find the value of c such that $a = e^c$. If we take $\log_e$ of both sides of this equation we find $c = \log_e(a)$. You will recall that this is an important result since it enables us to rewrite any function of the form $y(t) = y_0 a^{kt}$ in terms of the exponential function $y(t) = y_0 e^{ckt}$.

Question T15 Use your calculator to find x when (a) $e^x = 4.8$, (b) $e^x = 10$, (c) $e^x = 0.56$. □

Question T16 The number of organisms, N, in a certain population increases exponentially: $N = N_0 \exp(kt)$ where $k = 0.02\ h^{-1}$. If $N_0 = 1000$, how long does it take for the population to double, i.e. what is t when $N = 2000$? Without doing any further calculation, write down the time taken for N to increase from 2000 to 4000. □

Question T17 In a sample of radioactive material, the number, N, of polonium nuclei decays exponentially: $N = N_0 \exp(-\lambda t)$ where $\lambda = 0.0133\ s^{-1}$. How long does it take for the number of polonium nuclei to halve? □

Question T18 By finding a suitable value of k, rewrite the function $y(x) = 3^x$ in the form $y(x) = e^{kx}$. Check your answer by finding 3^x and $\exp(kx)$ when $x = 2$. □

3.3 Properties of logarithms

In this subsection we examine some further properties of logarithms that can be deduced from the relationships we have already discussed together with the rules for manipulating indices that were summarized in Equations 3 to 6.

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Products and quotients

First, let us look at the result of multiplying two numbers together where each is expressed as 10 raised to a power. Suppose that $x = 10^p$ and $y = 10^q$. By definition then, $\log_{10}(x) = p$ and $\log_{10}(y) = q$ (see Equation 15).

Now, $xy = 10^p \times 10^q = 10^{p+q}$

so $\log_{10}(xy) = \log_{10}(10^{p+q}) = p + q = \log_{10}(x) + \log_{10}(y)$

thus $\log_{10}(xy) = \log_{10}(x) + \log_{10}(y)$

Of course, instead of working to base 10, we could equally well have written x and y as powers of some other number a, and then taken logs to base a. If we had done so we would have found the following general result:

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad (22)$$

In Subsections 2.4 and 3.2 you saw that any positive number could be expressed as e raised to some power. In fact, any positive number can be expressed in terms of *any* given positive number (such as 10 or e) raised to a suitable power.

You can illustrate this rule with the aid of a calculator. For example, $\log_{10}(3) = 0.477$, $\log_{10}(7) = 0.845$ and $\log_{10}(21) = 1.322$ (all to three decimal places), i.e. $\log_{10}(21) = \log_{10}(3) + \log_{10}(7)$.

◆ Confirm that $\log_e(21) = \log_e(3) + \log_e(7)$.

◇ $\log_e(3) = 1.099$, $\log_e(7) = 1.946$, $\log_e(21) = 3.045$ (= 1.099 + 1.946). □

You have in fact already met another example of this rule. In Question T12, you saw that $\log_{10}(4.725 \times 10^n) = n + \log_{10}(4.725)$ — which follows from Equation 22 because $\log_{10}(10^n) = n$.

The next example demonstrates some properties of logs of quotients and reciprocals. (☞)

◆ (a) By writing $x = 10^p$ and $y = 10^q$, find an expression for $\log_{10}(x/y)$ in terms of $\log_{10}(x)$ and $\log_{10}(y)$.

(b) Use your answer to express $\log_{10}(1/x)$ in terms of $\log_{10}(x)$.

◇ (a) $\log_{10}(x/y) = \log_{10}(10^{p-q}) = p - q = \log_{10}(x) - \log_{10}(y)$.

(b) It follows that $\log_{10}(1/x) = \log_{10}(1) - \log_{10}(x)$. But, $\log_{10}(1) = 0$. So, $\log_{10}(1/x) = -\log_{10}(x)$. □

Reciprocals: recall that

$(a/b)^{-1} = 1/(a/b) = b/a$

$(a^x)^{-1} = 1/a^x = a^{-x}$

$(a^{-x})^{-1} = a^x$.

The example above suggests the following general rules, which apply to logs to any base a:

$$\log_a(x/y) = \log_a(x) - \log_a(y) \qquad (23)$$

$$\log_a(1/x) = -\log_a(x) \qquad (24)$$

◆ Check Equations 23 and 24 with the aid of your calculator. For example, try $x = 24$, $y = 8$, using (a) logs to base 10, and (b) logs to base e.

◇ (a) $\log_{10}(24) = 1.380$, $\log_{10}(8) = 0.903$; $\log_{10}(3) = 0.477 = \log_{10}(24) - \log_{10}(8)$

$\log_{10}(1/24) = \log_{10}(0.0417) = -1.380 = -\log_{10}(24)$.

(b) $\log_e(24) = 3.178$, $\log_e(8) = 2.079$; $\log_e(3) = 1.099 = \log_e(24) - \log_e(8)$

$\log_e(1/24) = \log_e(0.0417) = -3.178 = -\log_e(24)$. □

Powers and roots

Suppose we have some number $y = x^n$ where n is an integer, then

$$y = \underbrace{x \times x \times x \ldots \times x}_{n \text{ factors}}$$

It follows from Equation 24 that

$$\log_a(y) = \underbrace{\log_a(x) + \log_a(x) + \ldots + \log_a(x)}_{n \text{ terms}}$$

thus, $\log_a(x^n) = n \log_a(x)$.

Note that we are using logs to a general base a since Equation 24 applies for any positive value of a.

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This is indicative of a more general result that applies to any power of x, not just integer powers:

$$\log_a (x^b) = b \log_a (x) \qquad (25)$$

You can check Equation 25 with the aid of a calculator. For example, you can show that $\log_{10}(3^2) = \log_{10}(9) = 2\log_{10}(3)$; that $\log_{10}(49^{1/2}) = \log_{10}(7) = [\log_{10}(49)]/2$ and that $\log_{10}(4^{1.73}) = \log_{10}(11.004) = 1.73 \times \log_{10}(4)$.

You have already met some other examples of this rule. From Equation 24, $\log_a(1/x) = \log_a(x^{-1}) = -\log_a(x)$, i.e. $b = -1$ in Equation 25. Also, if $y/y_0 = e^{kt}$, then taking natural logs of both sides gives us $\log_e(y/y_0) = \log_e(e^{kt}) = kt \log_e(e) = kt$ (remember that $\log_e(e) = 1$) which is the expression we found in Subsection 3.2.

◆ By taking logs to base 10 of both sides of the equation, find the value of x when $2^x = 3 \times 5^x$.

◇ $x\log_{10}(2) = \log_{10}(3) + x\log_{10}(5)$. So $x[\log_{10}(2) - \log_{10}(5)] = \log_{10}(3)$, and hence $x\log_{10}(2/5) = \log_{10}(3)$, $x = \log_{10}(3)/\log_{10}(2/5) = -1.199$. □

As you will see in Subsection 3.4, Equations 22 to 25 enable logs to be used in tackling a wide variety of problems in physics. But before we end this subsection, we will extend our discussion and show how to convert between different bases for logarithmic functions.

Changing the base of logarithms

In Subsection 3.2 we derived the following expression involving $\log_a(x)$:

$$a^{\log_a(x)} = x \qquad \text{(Eqn 18)}$$

Now suppose we take logs to *another base*, b, of both sides:

$$\log_b\left(a^{\log_a(x)}\right) = \log_b(x) \qquad (26)$$

From Equation 25,

$$\log_a(x) \times \log_b(a) = \log_b(x) \qquad (27)$$

So we have a relationship between the log of x to two different bases — most usefully, it enables us to convert between the two most widely-used bases:

$$\log_{10}(x) \times \log_e(10) = \log_e(x) \qquad (28)$$

Note that we now know how to express a^x as e^{cx}, and $\log_a(x)$ as $\log_e(x)/c$, where $c = \log_e(a)$.

To three decimal places, $\log_e(10) = 2.303$, and so $\log_e(x) = 2.303 \times \log_{10}(x)$. Using your calculator, you can verify that Equation 28 describes the relationships between $\log_e(x)$ and $\log_{10}(x)$ for various values of x. But if you look at the answer to Question T14 you will see that the graph of $\log_e(x)$ can be obtained from the graph of $\log_{10}(x)$ by re-scaling the y-axis by a factor of $\log_e(10)$, just as Equation 28 implies.

Question T19 Use logs to base ten to find the value of x when $6 = 2^x$. □

Question T20 By finding $\log_{10}(5)$ and $\log_{10}(2)$ on a calculator, calculate $\log_2(5)$. □

3.4 Using logarithms in physics

We have already seen one use of logarithms in physics — given that $y = y_0 \exp(kt)$ we can find t in terms of y, i.e. $t = [\log_e(y/y_0)]/k$. (See Subsection 3.2, in particular Questions T16 and T17.) Now consider another problem related to exponential change. Suppose an experiment gives the results shown in Figure 17. Does this graph follow an exponential law? And if so, what is the value of k? It is hard to see whether or not Figure 17 really does represent an exponential function — you can measure gradients at various points, as we did in Subsection 2.2 (e.g. in Question T4), but this is time-consuming, and there are always uncertainties involved with drawing tangents.

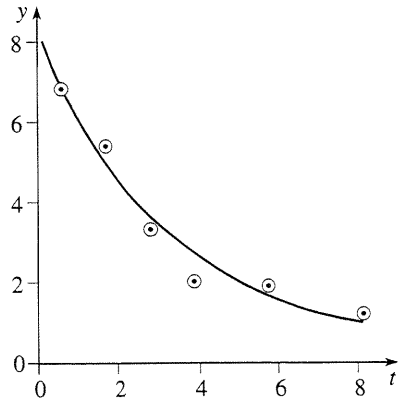


Figure 17 Experimental measurements of a quantity y that varies with time.

Let us consider an easier approach. If y does follow an exponential law, we must have $y = y_0 \exp(kt)$, and hence $\log_e(y/y_0) = kt$ (Equation 21). We can use Equation 23 to write $\log_e(y) - \log_e(y_0) = kt$, and hence

$$\log_e(y) = kt + \log_e(y_0) \qquad (29)$$

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If we recognize $\log_e(y)$ as a simple variable (call it Y if you like) then we see that Equation 29 has the general form of the equation of a straight line ($y = mx + c$), so if we plot $\log_e(y)$ against t, we should expect to get a straight line with gradient k that intersects the vertical axis at $\log_e(y_0)$. If the results do not follow an exponential law, the graph will not be a straight line.

power law

Note that we could get the same information by plotting $\log_e(y/y_0)$ — the choice of which to plot will in practice depend on whether it is easier to measure y itself, or as a fraction of its initial value.

Question T21 Figure 18 shows graphs of the results from two (hypothetical) experiments. Which graph shows that the relationship between y and t follows an exponential law, and what is the value of k in that case? □

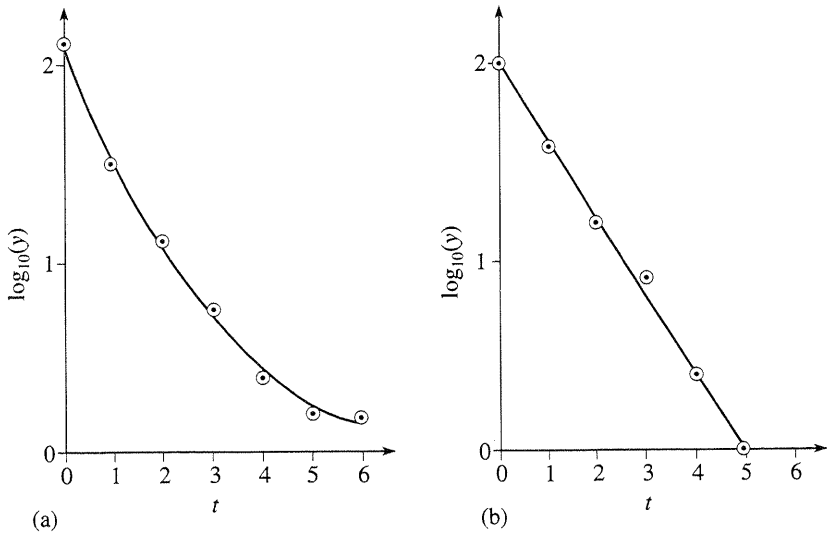


Figure 18 See Question T21.

Similar techniques can be used for variables that obey a **power law**, i.e. a relation of the form $y = kx^p$ (or $y = kt^p$) where the power, p, is some constant number. For example, suppose you think that variables F and r might obey a power law of the form

$$F = \frac{k}{r^2} \qquad (30)\ (☞)$$

Such a power law is known as an *inverse square law*. Coulomb's law for the electrostatic force between two charged particles, and Newton's law for the gravitational attraction of two masses, are both examples of an inverse square law.

where k is an unknown constant. How can you test your hypothesis experimentally? And how can you find the value of k? The properties of logarithms, as summarized in Equations 22–25, provide a way. First, write the expression $F = k/r^2$ in terms of $\log_{10}(F)$ and $\log_{10}(r)$, using Equation 23:

$$\log_{10}(F) = \log_{10}\left(\frac{k}{r^2}\right) = \log_{10}(k) - \log_{10}(r^2) \qquad (31)$$

Then, using Equation 25 we find:

$$\log_{10}(F) = -2\log_{10}(r) + \log_{10}(k) \qquad (32)$$

Alternatively, we could write $F = kr^{-2}$ and use Equation 22 to write $\log_{10}(F) = \log_{10}(k) + \log_{10}(r^{-2})$, then use Equation 25 to produce the same result.

◆ Suppose you were to plot a graph of $\log_{10}(F)$ against $\log_{10}(r)$. What would be its gradient and its intercept (☞) on the vertical axis?

◇ Equation 32 is an equation of the form $y = mx + c$, so the gradient would be -2 and the intercept $\log_{10}(k)$. □

The intercept is sometimes called the y-intercept.

Plotting a graph using logarithms thus enables you: to test the data to see whether they follow the expected exponential law, and to find the value of the constant k. Note that you do not have to use logs to base 10, since Equations 22–25 apply to logs to any base a. This technique can be summarized as follows:

if $y = kx^p$

then $\log_a(y) = p\log_a(x) + \log_a(k)$ (33)

so a graph of $\log_a(y)$ against $\log_a(x)$ has gradient p and intercept $\log_a(k)$. The logs may have any base, though bases 10 or e are generally used.

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Question T22 According to one of Kepler's laws of planetary motion, the orbital period, T, of a planet is related to its (average) distance, R, from the Sun: $T^2 = kR^3$. What would be the gradient of a graph of $\log_{10}(T)$ against $\log_{10}(R)$? What would be the gradient if log to base e was used rather than base 10? □

In Kepler's law, $k = (4\pi^2/GM)$ where G is the universal gravitational constant and M the mass of the Sun. See the *Glossary*.

Question T23 An experimenter measures the period of oscillation, T, of various masses m supported by a spring and suggests, on theoretical grounds, that $T \propto \sqrt{m}$. How could the experimental data be tested to see whether $T \propto \sqrt{m}$? □

4 Closing items

4.1 Module summary

1 If the graph of some function $y(t)$ is a straight line, then the *rate of change* of y is constant, and is equal to the gradient of the graph. If the graph is a curve, then the rate of change at any point is defined by the gradient of the *tangent* to the curve at that point.

2 If the rate of change of a variable is always proportional to the current value of that variable, then we can say that variable changes *exponentially*. If we represent the variable by y, and its rate of change by $\Delta y/\Delta t$, we can write $\Delta y/\Delta t = ky$, where a positive value of k characterizes *exponential growth*, and a negative value of k characterises *exponential decay*.

3 If $\Delta y/\Delta t = ky$, then $y(t) = y_0 \exp(kt)$, where the *exponential function* $\exp(kt) = e^{kt}$. The units of k are such that kt is a dimensionless quantity.

4 The following relationships are a consequence of the general properties of powers:

$$e^x e^y = e^{x+y}$$

$$(e^x)^y = e^{xy}$$

$$e^{-x} = \frac{1}{e^x}$$

$$e^{x/y} = (e^{1/y})^x$$

5 Exponential functions are not restricted to describing changes with time. For example, the function $\exp(kx)$ (with k such that kx is dimensionless) could describe how a quantity varies with position or any other quantity.

6 The constant, e, is given in terms of a *limit* by:

$$e = \lim_{m\to\infty}(1 + 1/m)^m = 2.718 \text{ (to three decimal places)}$$

7 The *logarithm to base a* of x is the power to which we must raise a to obtain x. So, if $x = a^y$, then $y = \log_a(x)$. The function $\log_a(x)$ is thus the inverse of the function a^x, hence $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$.

8 The most widely-used logarithmic functions have $a = 10$ and $a = e$.

9 Any function a^x can be written in terms of the exponential function: $a^x = \exp(cx)$, where $c = \log_e(a)$.

10 The following relationships arise from the definition of a logarithmic function and the rules for combining indices:

$$\log_a(xy) = \log_a(x) + \log_a(y) \quad \text{(Eqn 22)}$$

$$\log_a(x/y) = \log_a(x) - \log_a(y) \quad \text{(Eqn 23)}$$

$$\log_a(1/x) = -\log_a(x) \quad \text{(Eqn 24)}$$

$$\log_a(x^b) = b\log_a(x) \quad \text{(Eqn 25)}$$

11 The base (a or b) of any logarithmic function can be changed using the relationship

$$\log_a(x) \times \log_b(a) = \log_b(x) \quad \text{(Eqn 27)}$$

So $\log_a(x) = \log_e(x)/c$, where $c = \log_e(a)$.

12 If $y = y_0 \exp(kt)$, then $\log_e(y) = \log_e(y_0) + kt$, and a graph of $\log_e(y)$ against t has gradient k and intercept $\log_e(y_0)$.

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13 If y and x are related via a *power law*, $y = kx^p$, then

$$\log_a (y) = p \log_a (x) + \log_a (k) \qquad \text{(Eqn 33)}$$

and a graph of $\log_a (y)$ against $\log_a (x)$ has gradient p and intercept $\log_a (k)$.

4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Recognize examples of exponential change.

A3 Estimate the gradient at a point on a curved graph by drawing a tangent.

A4 Sketch a graph of $y = y_0 \exp(kt)$ and describe its significant features.

A5 Write down an expression for the rate of change of y, where $y = y_0 \exp(kt)$, for any given value of t or y.

A6 Recognize and interpret the common expression for the number e written in terms of a limit.

A7 Explain the relationships between the functions $\log_a (x)$, a^x and antilog (x) and know that many common applications of such functions use $a = \text{e}$ or $a = 10$.

A8 Sketch graphs of $y = \log_e x$ and $y = \log_{10} x$ and describe their significant features.

A9 Find the value of c such that $a^t = e^{ct}$.

A10 Manipulate and simplify expressions involving logarithmic and exponential functions of products, quotients and powers.

A11 Change the base of a logarithmic function and find the value of c such that $\log_a (x) = \log_e (x)/c$.

A12 Use a calculator to evaluate exponential functions, and logarithmic functions in base e and base 10.

A13 Use logarithmic functions to test whether experimental data obey an exponential law or a power law, and to find the values of unknown constants in such relationships.

4.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2* and *A3*) Figure 19 shows how a certain variable y changes with x. By drawing tangents, estimate the gradient $\Delta y/\Delta x$ at $y = 20$, $y = 40$ and $y = 60$. Are your answers consistent with y being an exponential function of x?

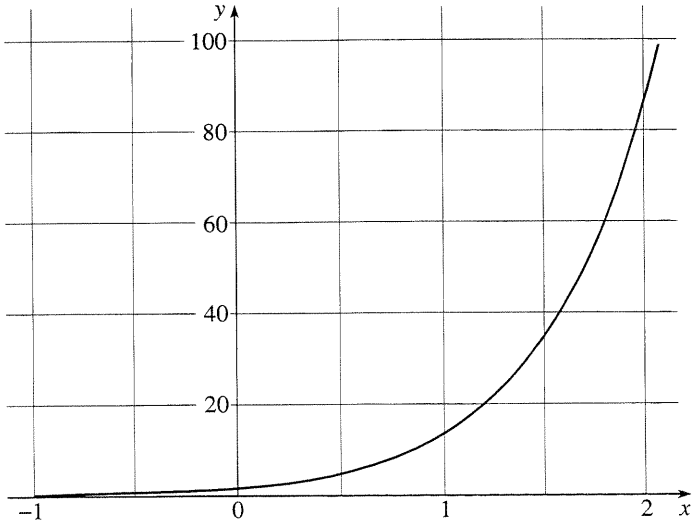


Figure 19 See Question E1.

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Question E2 (*A4*, *A5* and *A11*) For the function $y = y_0 \exp(kt)$ with $y_0 = 3$ and $k = 0.1$, (a) state a general expression for the gradient of the graph at any value of t, (b) calculate the gradient of the graph when $t = 0$ and when $t = 2$, and (c) sketch a graph of the function.

Question E3 (*A9*) Rewrite the equation $y = 10^{bx}$ in terms of the exponential function $\exp(x)$.

Question E4 (*A10*) Given that $\log_{10}(3) = 0.477$ and $\log_{10}(2) = 0.301$, evaluate the following without using a calculator: (a) $\log_{10}(300)$, (b) $\log_{10}(6)$, (c) $\log_{10}(60)$.

Question E5 (*A7* and *A10*) Without using your calculator, simplify (a) $\exp[\log_e(\pi)]$, (b) $\log_e(e^{\pi})$, and (c) $\log_{\pi}\left(\pi^{\sqrt{2}}\right)$.

Question E6 (*A11* and *A12*) Find $\log_e(\pi)$ and $\log_e(2)$ on your calculator and hence evaluate $\log_2(\pi)$.

Question E7 (*A12* and *A13*) The results of an (hypothetical) experiment are given in Table 5. Suppose the data obey a law of the form $T = kA^{\alpha}$. Find k and α by plotting a suitable graph.

Table 5 See Question E7.

A	T
1	0.70
3	1.46
6	2.31
10	3.25
20	5.16
30	6.76
45	8.86

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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5 Answers and comments

Fast track answers

F1 See Figure 4. At $x = 2$ the gradient of the tangent is 4. As this value has been derived from a graph, your value for the gradient may differ slightly from this.

F2 $\lim_{x\to\infty}$ is an instruction to evaluate the expression that follows as x tends to infinity. In this case $1/x$ tends to 0 as x tends to infinity.

F3 e = 2.718 ... (to three decimal places).

F4 The gradient is k.

F5 (a) $\log_e[(e^x)^y] = \log_e(e^{xy}) = xy$.

(b) This equation cannot be simplified.

(c) $\exp[(\log_e(x) + 2\log_e(y)] = \exp[(\log_e(x)] \times \exp[(\log_e(y^2)] = xy^2$.

(d) $\exp[2\log_e(x)] = \exp[\log_e(x^2)] = x^2$.

(e) $a^{\log_a(x)} = x$.

F6 $\log_{10}(P) = -a\log_{10}(f) + \log_{10}(k)$, thus, a graph of $\log_{10}(P)$ against $\log_{10}(f)$ is a straight line of gradient $-a$ and intercept $\log_{10}(k)$.

Ready to study answers

R1 (a) a^m, (b) $5^0 = 1$.

R2 The expression reduces to $a^{20} \times a^6 = a^{26}$. So $x = 26$.

R3 (a) $16^{-1/4} = (2^4)^{-1/4} = 2^{-1} = 1/2$.

(b) $16^{3/4} = (2^4)^{3/4} = 2^3 = 8$.

(c) $4^{5/2} = (2^2)^{5/2} = 2^5 = 32$.

(d) $27^{-2/3} = (3^3)^{-2/3} = 3^{-2} = 1/9$.

(e) $1/(3^{-2}) = 3^2 = 9$.

(If you had difficulty with Questions R1 to R3, refer to *powers* in the *Glossary*.)

R4 The *inverse function* $G(x)$ 'undoes' the effect of $F(x)$, so $G[F(x)] = x$, for any value of x in the *domain* of $F(x)$. If $F(x) = x^3$, then $G(x) = x^{1/3}$.

R5 The *graph* of $y = x^2$ is given in Figure 4. The approximate *solutions* are $x = 1.65$ and $x = -1.65$.

R6 The expressions in (a), (c), and (d) will give *straight-line* graphs. The gradients of these graphs are as follows: (a) m, (c) -1, (d) p.

R7 The 'rise' is 8 (= 13 − 5) and the 'run' is 2 (= 3 − 1) so the *gradient* (= rise/run) is 4.

(If you had difficulty with Questions R4 to R7, refer to *function*, *gradient* or *graph* in the *Glossary*.)

Text answers

T1 Number of disintegrations per second is given by $\lambda \times$ number of polonium nuclei = $0.0133\ s^{-1} \times 6.0 \times 10^{18} = 8.0 \times 10^{16}\ s^{-1}$. (In fact, this only represents the *average* number of decays per second, since nuclear decay is a statistical process in which it is not possible to predict the exact number expected in any particular second.)

T2 $k = -1/(RC)$. The rate of discharge is equal to the current, $I = Q/(RC)$, and Q is decreasing with time so k must be negative.

T3 For the tangent at $t = 10$ s, the 'rise' ≈ −20 and 'run' ≈ 28, so the gradient ≈ −20/28 ≈ −0.71, so the flow rate $\Delta V/\Delta t \approx -0.71$ litre s^{-1}. The line passing through $t = 0$ s, $V = 30$ litre, and $t = 15$ s, $V = 0$ litre, is an approximate tangent to the curve at $t = 5$ s, so $\Delta V/\Delta t \approx -2$ litre s^{-1}. (The exact value you find will depend on how you draw the tangent.)

T4 The accurate values for the currents determined for the tangents are 0.5 A, 0.25 A and 0.125 A, i.e. $\Delta Q/\Delta t \approx 0.25\ s^{-1} \times Q$. (Your values should approximate these.) Therefore the decay is exponential. (Of course, rather imprecise measurements made at a few points on a graph do not really verify that the graph is exponential, but at least they haven't given any evidence to the contrary, or rather, they shouldn't have done so!)

T5 One way would be to show that the gradient is not proportional to the value of V. For instance, you could measure the gradient at $x = 1$ (where $V = 8$) and at $x = 2$ (where $V = 2$) and show that the gradients at those two points, −16 and −2, respectively, are not in the ratio 8 : 2.

T6 $e^2 = 7.389$, $e^3 = 20.09$, $e^{1.43} = 4.179$, $e^{-1} = 0.3679$ (all to four significant figures), $e^0 = 1$ ($a^0 = 1$ for all values of a).

T7 $k = -1/RC$ (see Answer T2) and so $Q(t) = Q_0 e^{-t/RC}$. Note that this could also be written as $Q(t) = Q_0 \exp(-t/RC)$

T8 $N_0 = 20 \times 10^{10}$ ($= 2 \times 10^{11}$). When $N = 10 \times 10^{10}$, the gradient $\Delta N/\Delta t = -5 \times 10^4\ s^{-1}$, so

$k = (\Delta N/\Delta t)/N = -5 \times 10^{-7}\ s^{-1}$.

T9 The completed table is given in Table 6.

Table 6 See Answer T9.

m	a
2	2.2500
5	2.4883
10	2.5937
10^2	2.7048
10^3	2.7169
10^4	2.7181
10^5	2.7183

T10 (a) $100 = 10^2$, so $\log_{10}(100) = 2$.

(b) $1000 = 10^3$, so $\log_{10}(1000) = 3$.

(c) $0.1 = 10^{-1}$, so $\log_{10}(0.1) = -1$.

(d) $0.001 = 10^{-3}$, so $\log_{10}(0.001) = -3$.

(e) $10^{1/2} = 10^{0.5}$, so $\log_{10}(10^{1/2}) = 0.5$.

(f) $10 = 10^1$, so $\log_{10}(10) = 1$.

(g) $1 = 10^0$, so $\log_{10}(1) = 0$.

(h) $\log_{10}(10^{1.52}) = 1.52$.

T11 (a) If $10^x = 6.8$, then $x = \log_{10}(6.8) = 0.8325$.

(b) $10^x = 537$, $x = \log_{10}(537) = 2.730$.

(c) $10^x = 0.34$, $x = \log_{10}(0.34) = -0.4685$.

(All to four significant figures.)

T12 (a) $\log_{10}(4.725) = 0.6744$.

(b) $\log_{10}(47.25) = 1.6744$.

(c) $\log_{10}(472.5) = 2.6744$.

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(d) A pattern has emerged — namely, $\log_{10}(4.725 \times 10^n) = n + \log_{10}(4.725)$. Therefore $\log_{10}(4725) = 3.6744$ and $\log_{10}(4.725 \times 10^7) = 7.6744$. (This pattern is discussed further in Subsection 3.3.)

T13 $\log_e(1) = 0$, since $e^0 = 1$ ($a^0 = 1$ for *any* value of a). $\log_e(e) = 1$, since $e^1 = e$.

T14 Figure 20 shows the graph of the function $\log_e(x)$ drawn on the same axes as $\log_{10}(x)$. The graph of $\log_e(10)$ is similar in shape to Figure 16b. Like $\log_{10}(x)$, it cuts the x-axis at $x = 1$. The rest of the curve can be estimated by drawing a curve through $\log_e(10) \approx 2.303$, $\log_e(e) = 1$ ($e \approx 2.718$) and $\log_e(4.5) \approx 1.5$ (see the seeded question above Question T13). (The exact relationship between $\log_{10}(x)$ and $\log_e(x)$ is discussed in Subsection 3.3.)

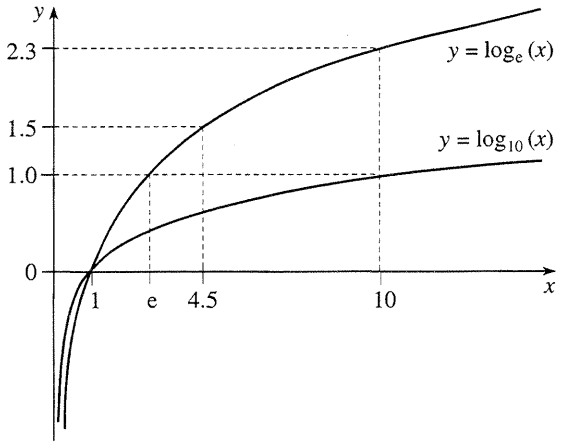


Figure 20 See Answer T14.

T15 (a) $e^x = 4.8$, $x = \log_e(4.8) = 1.569$.

(b) $e^x = 10$, $x = \log_e(10) = 2.303$.

(c) $e^x = 0.56$, $x = \log_e(0.56) = -0.5798$. (All to four significant figures.)

T16 $N = N_0 \exp(kt)$. Dividing both sides by N_0, and using the fact that $N/N_0 = 2$, gives us $2 = \exp(kt)$ so $\log_e(2) = kt$ and hence $t = \log_e(2)/k = 0.693/(0.02\ h^{-1}) = 34.7$ h. This applies both when $N_0 = 1000$ and $N = 2000$, and when $N_0 = 2000$ and $N = 4000$ — the time for the population to double is independent of the initial number N_0.

T17 $N = N_0 \exp(-\lambda t)$. Dividing both sides by N_0, gives us $N/N_0 = \exp(-\lambda t)$ where $N/N_0 = 1/2$ so that $1/2 = \exp(-\lambda t)$ and therefore (taking logs of both sides) we find $\log_e(1/2) = -\lambda t$ so that

$$t = [\log_e(1/2)]/(-\lambda) = 0.693/(0.0133\ s^{-1}) = 52.11\ s.$$

T18 $e^k = 3$, so $k = \log_e(3) \approx 1.099$ (to four significant figures). $y = \exp(1.099x)$. $3^2 = 9$, and $\exp(1.099 \times 2) \approx 9.007 \approx 9$. (The agreement becomes even better if k is taken to more significant figures.)

T19 $\log_{10}(6) = x \log_{10}(2)$, so $x = \log_{10}(6)/\log_{10}(2) \approx 2.585$.

T20 From Equation 27, with $b = 10$, $a = 2$ and $x = 5$: $\log_2(5) \times \log_{10}(2) = \log_{10}(5)$ and so

$$\log_2(5) = \log_{10}(5)/\log_{10}(2) \approx 2.322.$$

T21 Figure 18b shows exponential decay, since the graph of $\log_{10}(y)$ plotted against t is a straight line. The gradient of this graph is $-2/5 = -0.4$. Thus the exponential law is $y = y_0 e^{-0.4t}$.

T22 Taking $\log_{10}$ of both sides gives us,

$$2 \log_{10}(T) = \log_{10}(k) + 3 \log_{10}(R)$$

so $$\log_{10}(T) = 3 \log_{10}(R)/2 + \log_{10}(k)/2$$

The gradient would be 3/2, irrespective of the base of the logs, since Equations 22–25 apply to logs to any base.

T23 If $T = k\sqrt{m}$, then $\log_{10}(T) = \log_{10}(k) + \log_{10}(\sqrt{m}) = \log_{10}(k) + [\log_{10}(m)]/2$, so a graph of $\log_{10}(T)$ against $\log_{10}(m)$ would have gradient 1/2. (Note that logs to any base could be used and the gradient would still be the same.)

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Exit test answers

E1 The measured gradients are approximately 45, 90 and 135 (the exact values depend on how you draw the tangents). These results are consistent with y being an exponential function of x, since the gradient is proportional to y ($\Delta y/\Delta x \approx 2.25y$).
(Reread Subsections 2.1 and 2.2 if you had difficulty with this question.)

E2 (a) Gradient $\Delta y/\Delta t = ky = ky_0 \exp(kt) = 0.3 \exp(0.1t)$.

(b) When $t = 0$, $\Delta y/\Delta t = ky_0 = 0.3$; when $t = 2$, $\Delta y/\Delta t = 0.3 \exp(0.2) \approx 0.3664$.

(c) Your sketch should resemble Figure 10, with the curve intercepting the y-axis at $y = 3$.

(Reread Subsections 2.3 and 2.4 if you had difficulty with this question.)

E3 We need a number k such that $10^{bx} = e^{kx}$, i.e. $10^b = e^k$. If we take $\log_e$ of both sides, we find $b \log_e(10) = k$, so $y = \exp[bx \log_e(10)]$.

(Reread Subsection 3.2 if you had difficulty with this question.)

E4 (a) $\log_{10}(300) = \log_{10}(3 \times 100) = \log_{10}(3) + \log_{10}(100)$ and $\log_{10}(100) = 2$, therefore $\log_{10}(300) = 0.477 + 2 = 2.477$.

(b) $\log_{10}(6) = \log_{10}(3 \times 2) = \log_{10}(3) + \log_{10}(2)$, therefore $\log_{10}(6) = 0.477 + 0.301 = 0.778$.

(c) $\log_{10}(60) = \log(6 \times 10) = \log(6) + \log(10)$ and $\log(10) = 1$, therefore $\log_{10}(60) = 0.778 + 1 = 1.778$.

(Reread Subsections 3.1 and 3.3 if you had difficulty with this question.)

E5 (a) $\exp[\log_e(\pi)] = \pi$,
(b) $\log_e(e^{\pi}) = \pi$,
(c) $\log_{\pi}\left(\pi^{\sqrt{2}}\right) = \sqrt{2}$.

(Reread Subsections 3.1, 3.2 and 3.3 if you had difficulty with this question.)

E6 $$\log_2(\pi) = \frac{\log_e(\pi)}{\log_e(2)} = \frac{1.145}{0.693} = 1.65$$

(Reread Subsection 3.3 if you had difficulty with this question.)

E7 Taking logs to base 10 of both sides of this equation we get $\log_{10}(T) = \log_{10}(k) + \alpha \log_{10}(A)$. The values are given in Table 7. This is a straight line of the form $y = mx + c$, therefore if $\log_{10}(T)$ is plotted against $\log_{10}(A)$ (as in Figure 21), the intercept on the vertical axis will give us $\log_{10}(k) = -0.16$ so $k = 0.7$, and the gradient gives us $\alpha \approx 0.67$. (You could plot $\log_e(T)$ against $\log_e(A)$, whichwould give a straight line with the same gradient α, but the intercept would be $(k) = -0.36$, giving $k = 0.7$ as before.)
(Reread Subsections 3.3 and 3.4 if you had difficulty with this question.)

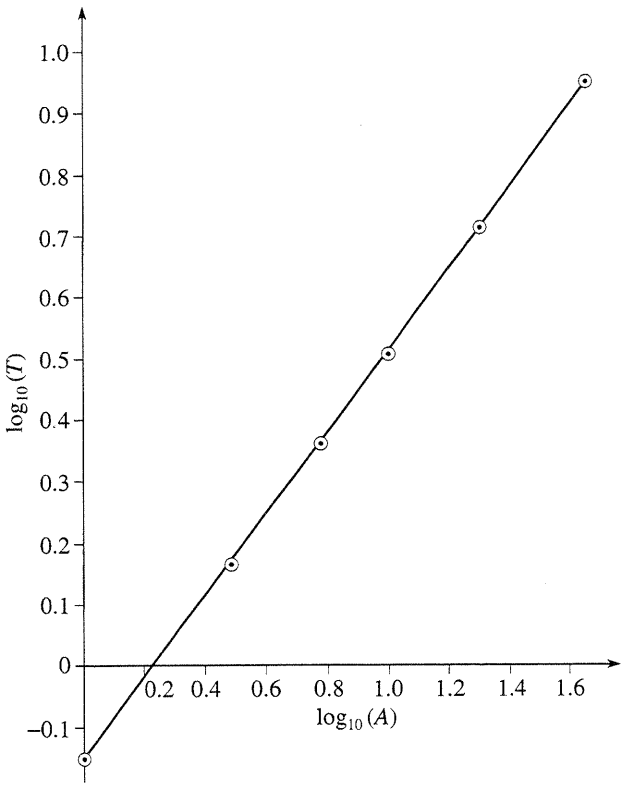


Figure 21 See Answer E7.

Table 7 See Answer E7.

A	T	$\log_{10}(A)$	$\log_{10}(T)$
1	0.70	0.00	−0.155
3	1.46	0.48	0.164
6	2.31	0.79	0.364
10	3.25	1.00	0.512
20	5.16	1.30	0.713
30	6.76	1.48	0.830
45	8.86	1.65	0.948

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