Solving equations

Module M1.4

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1 Opening items

1.1 Module introduction

When we do anything in physics, there comes a point when we have to produce numbers; either as the result of an experimental measurement, or as the prediction of a theory, or both. Almost always this means having to do some calculations; not just adding numbers, or multiplying them, but manipulating them in some way so as to produce an answer; often this involves *solving* equations. In this module we will look at how to solve the simplest (and commonest) equations, and outline some of the more generally applicable ways in which we can solve more difficult problems.

Initially, we will look at solving *linear equations* (in Section 2), i.e. equations in which the independent variable appears only as the first power. In Section 3 the discussion is extended to the solution of *quadratic equations* by various methods, e.g. *factorizing*, *completing the square* and using the quadratic equation formula. Finally, in Section 4 graphical and numerical procedures for solving *polynomial equations* are described.

The first thing you must do before starting this module is to **PUT YOUR CALCULATOR AWAY**. You will not need it for most of this module, and you will be told when you do need it. Calculators are very useful for doing calculations to great accuracy, but they can stop you thinking about the important features of calculations; they are often a good means of making mistakes quickly!

Study comment Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

1.2 Fast track questions

Study comment Can you answer the following *Fast track questions*? The answers are given in Section 6. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 5.1) and the *Achievements* listed in Subsection 5.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 5.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

Question F1 Find the value of x that satisfies the following equation:

$$\frac{x-1}{x+1} = 2$$

Question F2 Solve the following equations:

$$x + 2y = 4$$

$$2x - y = 3$$

Question F3 Find the roots of the following equation

$$x^2 + x - 1 = 0$$

leaving the square roots in the answer. How can you verify that your answers are right?

1.3 Ready to study?

Study comment In order to study this module you will need to understand the following terms: *constant*, *equation*, *function*, *graph*, *powers*, *square root* and *variable*. If you are uncertain about any of these terms you should review them now by referring to the *Glossary*,

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which will indicate where in *FLAP* they are developed. In addition, you will need to be able to *expand*, *simplify* and *evaluate* simple *algebraic expressions* that involve *brackets*. You will also need to be able to *sketch* and *plot* the graphs of simple functions. If you are uncertain about your ability to perform these operations, you should again refer to the *Glossary* for further information. The following *Ready to study questions* will help you to check that you have the required skills and knowledge before embarking on this module.

unknown variable
linear equations
linear form
solving the equation
solution
root
soluble
insoluble

You will find the answers to these questions in Section 6.

Question R1 Expand the following expressions:

(a) $(a-2b)(a+2b)$ (b) $(x+3)(x-4)$ (c) $(p-3)^2$

Question R2 Simplify the following expression:

$2(x^2+3x-2)-(2x^2-6x-4)$

Question R3 If $f(x)=3x+2$, write down expressions for $f(2y)$ and $f(y-1)$.

Question R4 What are the square roots of x^4?

Question R5 Sketch graphs of the following functions from $x=-1$ to $x=3$.

$f(x)=2x-1$ and $g(x)=x^2-3x+2$

Sketch means draw a rough graph, showing the salient features — it does not mean plot the graph accurately.

Question R6 Rewrite, with a single denominator, the expression

$$\frac{1}{1-x}-\frac{1}{1+x}$$

The *denominator* is the number or expression on the bottom of a fraction.

---

2 Solving linear equations

When we attempt to solve an equation for an **unknown variable**, x say, we aim to find an expression which looks something like this

$$x = ??? \qquad (1)$$

where the right-hand side does not involve x and is something we can calculate immediately. When we have reached this point the mathematics is done and arithmetic takes over.

In this section we shall look at **linear equations**, which only involve the first power of the unknown variable, x, rather than x^2 or any higher power. In other words we will look at equations such as

$$3x+2=5x-2$$

which are said to be in **linear form**. We shall attempt to find the value (or values) of the unknown variable for which the equation is *true*. In the case of the above equation it is easy to see that the value $x=2$ makes the equation true since for this value the left-hand side becomes $3\times 2+2=8$, while the right-hand side is $5\times 2-2=8$. We call this process **solving the equation**, and we speak of finding the **solution**, or finding the **root** of the equation. Later we will extend these notions to more than one equation. Equations that can be solved are sometimes said to be **soluble**, while those that can't are **insoluble**.

2.1 Simple linear equations; solving by rearrangement

Suppose you set your stop-watch to zero as you start cycling along a road at 10 mph. After you have travelled 15 miles you stop for a rest. How long has this taken you?

The distance you have travelled after a period of time t is

$$d = vt$$

distance = speed × time

where v is your speed. In this example we know what d and v are, and we have to find t. Dividing both sides of the above equation by v we see that

$$t=\frac{d}{v}=\left(\frac{15}{10}\right)\text{hours}$$

Thus we have reached the point at which arithmetic takes over and the final answer is 1.5 hours. This simple problem was solved by rearranging the equation.

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Now for a slightly more complicated problem. Forget for the moment that $d = 15$ miles and $v = 10$ mph, and think about the following problem in purely algebraic terms. Suppose that having cycled a distance d at speed v you then rested for a time t_R, and that following your rest you set out again on a second ride at speed v. If you kept cycling until your total journey time (including the rest) was T, what distance would you have covered in that time?

The time spent on the second ride is found by subtracting the rest time and the time of the first ride from the total journey time, so it is $(T - d/v - t_R)$; and the total distance D covered in both rides is just the sum of the distances covered in each ride, so

$$D = d + v(T - d/v - t_R) \qquad (2)$$

It is easy to rearrange this equation to find an expression for T in terms of the total distance travelled, the speed, the distance d and the rest time. The form of this expression may already be obvious to you, but being systematic and spelling everything out in detail we can start by trying to isolate the bracket that contains T. To do this note that from Equation 2

$$D - d = v(T - d/v - t_R)$$

so that $(T - d/v - t_R) = (D - d)/v$

Thus, $T = d/v + t_R + (D - d)/v = d/v + t_R + D/v - d/v$

You can see that d/v cancels out on the right-hand side of the last equation, so

$$T = t_R + D/v \qquad (3)$$

You might have realized that we could have moved directly to Equation 3 by equating the total time T to the sum of the rest time t_R and the total cycling time D/v.

Now we can substitute in some numbers; suppose the total distance travelled is 35 miles and $v = 15$ mph; if the rest time was half an hour then $t_R = 0.5$ hours and

$$T = \left(\frac{1}{2} + \frac{35}{10}\right)\text{hours} = (0.5 + 3.5)\,\text{hours} = 4\,\text{hours}$$

You may be wondering why we didn't put these numbers in at the beginning. There are three reasons for this; first, if you look again at Equation 3 you will see that the variable d has disappeared, so less calculation is needed than if we had used numbers all the way through. Second, having worked out the general equation, we can change the distances, or the speed, and still find the time taken without having to go through the whole calculation again. Finally, it is often easier to check a calculation that involves algebraic variables rather than numerical values.

◆ Suppose you cycled non-stop for 36 miles at 12 mph. How long would this have taken?

◇ We just need to substitute $t_R = 0$, $D = 36$ miles, $v = 12$ mph in Equation 3 to obtain

$T = (0 + 36/12)$ hours = 3 hours □

This example provides us with a rule that applies throughout mathematics and physics. To avoid unnecessary effort and to reduce the chance of mistakes:

Don't introduce numbers until it is essential; leave all the arithmetic until the end.

Let us now return to the business of solving equations. The following method applies to any equation in which the unknown variable appears to the first power only, and only in the numerator. The procedure in general is much like the one described above; for brevity let us suppose the unknown variable is x.

The *numerator* is the number or expression on the top of a fraction.

To solve a linear equation, e.g. $3(x - 1) = 2x + 5(2 - x)$

(a) if x appears in more than one bracket, expand the brackets; $3x - 3 = 2x + 10 - 5x$

(b) collect together all the terms involving x on one side of the equation and the remaining terms on the other; $3x + 5x - 2x = 10 + 3$

(c) combine the **coefficients** of x as far as possible (i.e. simplify all the terms multiplying x), and then simplify all the remaining terms; $6x = 13$

(d) you should now have an equation that can be written in the form of Equation 1 with no more than one more step. $x = \frac{13}{6}$

Always check that your answers satisfy the original equations and that they make sense.

coefficient

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With practice it is possible to condense this process by combining the steps, but be warned, doing too many steps in your head is likely to lead to mistakes. Try the next four questions, all of which can be solved by the above method.

Question T1 Solve the equation $2(x + 1) = x - 5$. □

You will find the answers to all *Text questions* in Section 6.

Question T2 Solve the equation

$$(z - 2) + 3(2z + 1) - (5z - 7) = 0 \quad \square$$

Question T3 If $a = 1$ and $b = 4$, find the value of p that satisfies

$$p - a = 3p + 2b \quad \square$$

Question T4 If $a = 4$ and $b = -2$, find the value of p that satisfies

$$p - a = 3p + 2b \quad \square$$

It is sometimes possible to solve what appear to be more complicated problems in the way described above. The general principle is to reduce more complicated equations to a form that we know how to solve. For example,

$$\frac{x-1}{x+2} = 3 \tag{4}$$

can be rewritten, if both sides are multiplied by $(x + 2)$, as

$$x - 1 = 3(x + 2)$$

and then treated as before. However, it is not always possible to reduce such an equation to linear form.

◆ Try to simplify the following:

$$\frac{x-1}{x+2} = \frac{3}{x}$$

◇ If we multiply by $(x + 2)$ and by x, to obtain

$$x(x - 1) = 3(x + 2)$$

This is not a linear equation in x.

we are left with a term in x^2, which cannot be removed. We shall see later that such equations can be solved, but for the moment we stop here. □

There are many equations that appear at first sight to be beyond the realm of linear equations, but which can be simplified to an unexpected degree. Look, for example, at the following

$$\frac{x-1}{x+2} = \frac{x+3}{x-1}$$

If we multiply both sides of the equation by $(x + 2)(x - 1)$ we have

$$(x - 1)^2 = (x + 3)(x + 2)$$

which implies

$$x^2 - 2x + 1 = x^2 + 5x + 6$$

i.e. $7x = -5$

so $x = -5/7$

However, events like this should be regarded as just a lucky accident, and are not to be relied on — though it is legitimate to hope!

Question T5 Which of the following equations can be reduced to a linear form? Solve those which can.

(a) $\dfrac{1}{x-1} = 6$ (b) $x^2 - 1 = \dfrac{2}{x+1}$ (c) $x^2 + 2 = x(x - 3)$

(d) $\dfrac{(x+2)(x-2)}{x^2-4} = x$ □

In real life it is not always obvious (as we can see from what we have just discussed) which equations can be reduced to the form of Equation 1. There is no general method which will allow us to take an arbitrary equation and say 'yes, this

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can be solved like this', and then write down the answer. Some equations can be solved easily, some with more difficulty, and some are impossible to solve. One of the advantages that comes from practice is the ability to recognize the category to which a particular equation belongs, so that you don't waste time trying to solve something insoluble, and you can go straight to the right method for something soluble.

simultaneous linear equations
substitution

2.2 Equations linear in a function of x

There is a slight and fairly obvious generalization of the linear equation that arises if the variable is, say x^2, rather than x. For example,

$$x^2 + 4 = 2x^2 - 12$$

can be solved for the unknown variable x^2 to give $x^2 = 16$, i.e. $x = \pm 4$. A similar method can be used in the following example.

◆ Solve the following equation $1/(x-1) + 2 = 2/(x-1) - 4$.

◇ In this case we initially solve for the unknown variable $1/(x-1)$.

Thus $1/(x-1) - 2/(x-1) = -4 - 2$

so $1/(x-1) = 6$

We then solve this equation for x, as in Question T5(a), to obtain

$x = 7/6$ □

We could have reduced the chance of a mistake in copying, by substituting $y = 1/(x-1)$ at the start of this question. This trick is well worth remembering — try it on the next exercise.

Question T6 Solve the following equation:

$$2\left(\frac{x^2+1}{x^2-1}+1\right) = \frac{x^2+1}{x^2-1} + 5 \quad \left(\text{hint: put } y = \frac{x^2+1}{x^2-1}\right) \quad \square$$

2.3 Simultaneous linear equations

Let us reconsider the cycling problem discussed in Subsection 2.1. Again suppose that you cycled along a road at a steady 10 mph. One hour after you left, a friend set off in a car at a steady 30 mph to catch you up. Where and when will you meet?

Let us take the distance to the meeting-point to be d_m, and the time on your stop-watch to be t_m when you meet. To keep things as general as possible, take the time at which the car starts to be T_0, and the speeds of the bicycle and the car to be v and V, respectively. At the instant you meet *you will each have travelled the same distance*, and so we have two equations analogous to Equation 2:

distance travelled by you = your speed × time spent cycling

i.e. $d_m = vt_m$ (5)

and

distance travelled by your friend = your friend's speed × time spent driving

i.e. $d_m = V(t_m - T_0)$

so $d_m = Vt_m - VT_0$ (6)

We know the values of v and V in these equations, and we can calculate VT_0; but we do not know the distance d_m or the time t_m, and we cannot work them out from either Equation 5 or Equation 6 alone (☞). Equations 5 and 6 are called **simultaneous linear equations** since they must be solved together if they are to be solved at all.

Equations 5 and 6 each contain *partial* information about the problem; but we need *all* the information contained in *both* equations to solve the problem.

The basic method of solving a pair of equations of this kind involves **substitution**. This means that we find an expression for one of the variables using one equation, and we then substitute that expression for the selected variable in the other equation.

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For instance, Equation 5 tells us that we can replace d_m by vt_m; making this substitution in Equation 6 gives

$$vt_m = Vt_m - VT_0$$

i.e. $vt_m - Vt_m = -VT_0$

i.e. $(v - V)t_m = -VT_0$

so $$t_m = \frac{-VT_0}{v - V} = \frac{VT_0}{V - v}$$

This is part of the solution we seek since it expresses one of the unknown quantities (t_m) in terms of known quantities. (It also makes good sense physically. Notice that t_m increases as T_0 increases, which we should expect, because the later your friend leaves, the longer it will be before you meet. Furthermore, if $V = v$ the bicycle and the car are travelling at the same speed, so the denominator becomes zero, t_m becomes infinite, and you never meet — which is again what we expect.) All that remains now is to find d_m, which we can do directly by substituting for t_m in Equation 5:

$$d_m = vVT_0/(V - v)$$

Now we can put in the numerical values $v = 10$ mph, $V = 30$ mph, $T_0 = 1$ h, and find

$$t_m = \left(\frac{30 \times 1}{30 - 10}\right) \text{hours} = 1.5 \text{ hours} \quad \text{and} \quad d_m = (10 \times 1.5) \text{ miles} = 15 \text{ miles}$$

Finally, we should check to see that these values do actually satisfy the original equations. We have used Equation 5 directly to find d_m, but Equation 6 correctly gives

$$15 \text{ miles} = 30 \text{ mph} \times (1.5 - 1) \text{ h} = 15 \text{ miles}$$

You will have noticed that we have put great emphasis on checking the answers; many wrong answers are due to bad arithmetic and algebraic slips. *Always check that your answers satisfy the original equations and that they make sense.* For example, an answer $t_m = -3$ hours would not be sensible.

Notice that the problem we have just solved had *two* equations and *two* unknown variables t_m and d_m. In general, a pair of simultaneous linear equations may look more like this:

$$x + 2y = 8 \tag{7}$$

$$2x - 3y = -5 \tag{8}$$

and we may be required to solve these equations for the two variables x and y.

We can express x in terms of y (using Equation 7)

$$x = -2y + 8 \tag{9}$$

and then substitute for x in Equation 8 to obtain

$$2(-2y + 8) - 3y = -5$$

so that $-4y + 16 - 3y = -5$

i.e. $7y = 21$

giving $y = 3$

If we substitute this value for y in Equation 9 we obtain

$$x = (-2 \times 3) + 8 = 2$$

Now we should check to see that these answers satisfy the original equations. We have just used Equation 7 (in the form of Equation 9) to find x, so we check by substituting both values into the other equation, Equation 8, and we find

$$(2 \times 2) - (3 \times 3) = 4 - 9 = -5$$

which is correct. Notice that for checking we always use the original equation (before we have had a chance to make any mistake), and we don't use the equation that has just been used to find one of the variables.

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elimination

You can apply this same procedure to any pair of linear equations. However, if the coefficients are not the simple integers we have here, the algebra may soon start to look complicated, and it is useful to have some sort of drill for the process of solution. Look again at the equations we have just solved.

$$x + 2y = 8 \qquad \text{(Eqn 7)}$$

$$2x - 3y = -5 \qquad \text{(Eqn 8)}$$

To get an equation involving just y, multiply the equations by suitable factors so that both have the same coefficient of x — in this case we multiply both sides of Equation 7 by 2 and leave Equation 8 unchanged:

$$2x + 4y = 16$$

$$2x - 3y = -5 \qquad \text{(Eqn 8)}$$

Now subtract the second equation from the first, to get

$$(2x + 4y) - (2x - 3y) = 16 - (-5)$$

i.e. $0x + 7y = 21$

so, as before

$$y = 3$$

This process is called **elimination**, we speak of *eliminating* x from the equations, though it's really just a systematic way of implementing substitution.

We now proceed as before and use Equation 7 to give $x = 2$ again. After this we should use Equation 8 to carry out the same check.

Alternatively, we could have eliminated y from Equations 7 and 8; and the simplest way to do that is to multiply Equation 7 by 3, and Equation 8 by 2 to obtain

$$3x + 6y = 24$$

$$4x - 6y = -10$$

Now we may *add* the above equations to obtain

$$7x + 0y = 14$$

which gives $x = 2$

We can then use either Equation 7 or Equation 8 to find y, and the other to check the answer.

So we have seen three alternative methods of solving Equations 7 and 8:

- substituting for x (found from one equation) in terms of y (in the other)
- eliminating x
- eliminating y.

A fourth method of solving Equations 7 and 8 would have been to substitute for y in terms of x.

In general we try to choose the method that is most convenient.

Question T7 Solve the following pairs of equations:

(a) $5x + 2y = 0$ $\quad$ (b) $4x + y = 10$

$2x + 5y = 21$ $\quad$ $3x - 2y = 13$ □

We can also work out the general solution to a pair of simultaneous equations using symbols rather than numbers. For example, for the pair of equations

Try this for yourself by substituting for x (from one equation) in terms of y (in the other).

$$ax + by = p \qquad (10)$$

$$cx + dy = q \qquad (11)$$

the solution (provided $ad - bc \neq 0$) is

$$x = (dp - bq)/D \qquad (12)$$

$$y = (aq - cp)/D \qquad (13)$$

where $D = ad - bc$ (14)

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This general solution is not particularly helpful, and certainly not worth remembering. However, you may meet it again when you study the theory of sets of three or more simultaneous equations.

independent
inconsistent

Question T8 Show that Equations 12–14 provide the correct solution to Equations 7 and 8. □

Before leaving the subject of simultaneous linear equations it's worth noting that such equations are not always soluble. Here are two situations where Equations 10 and 11 do not have a solution.

Case 1

This is the case in which the equations are not **independent** — i.e. one equation is a multiple of the other, so the second equation provides no new information; an example would be the pair of equations

$$x + 2y = 8$$

and $$2x + 4y = 16$$

If we try to eliminate x from these equations we shall just end up with $0 = 0$, a statement which is true, but not much use.

Case 2

This is the case in which the equations are **inconsistent** — they cannot both be true at the same time. For example,

$$x + 2y = 8$$

and $$2x + 4y = 15$$

If we try to solve this pair of equations we shall find $1 = 0$, which is neither useful nor true.

The first case occurs when we don't have enough information to solve a problem, the second when some of the information is wrong or inconsistent. Note that in either case the general solution of Equations 12–14 does not apply since $D = 0$.

In this section we have been concerned with *two* linear equations, and *two* unknowns; however, there are many occasions when one has to deal with three or more unknowns. In general:

> If there are n unknowns, then n consistent independent linear equations are needed to solve the equations for the unknowns.

If we are given more than n consistent linear equations, they cannot all be independent, so we are free to select n of them that *are* independent and use those to find the n unknowns. Finally, it should be noted that when seeking a set of three or more *independent* linear equations it is not enough to require that no equation should be a *multiple* of any other; rather we have to demand that none of the equations can be expressed as a *sum of multiples* of the others. It is actually fairly easy to determine whether or not a given set of equations satisfies this requirement, but it requires techniques beyond the scope of this module.

2.4 Graphical solutions

There is one other way of solving a pair of simultaneous equations that is worth mentioning — a graphical method. Let us consider Equations 7 and 8 yet again. If we write these in the equivalent form

$$y = \frac{8 - x}{2}$$

$$y = \frac{2x + 5}{3}$$

we can plot each of them as a straight-line graph, as in Figure 1. On the first line, $y = (8 - x)/2$, we have all the points (x, y) which make this equation a *true* statement. Similarly, on the second line, $y = (2x + 5)/3$, we have all the points (x, y) which make *this* equation a *true* statement. If *both* statements are true for a particular point (x, y) then this point must be where the two lines intersect, for it is only there that both equations are true simultaneously.

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This method can be used to find an approximate solution to any pair of equations (linear or not) and it is often a good starting point, particularly if you have access to a graph-drawing computer or calculator.

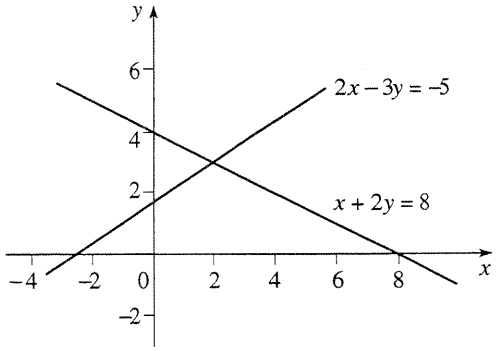


Figure 1 Graphical solution of Equations 7 and 8.

Question T9 Use a graphical method to find approximate solutions of the simultaneous equations

$$x + 2y = k \quad \text{and} \quad 2x - 3y = -5$$

for $k = 2$, 4, 6 and 8. (You can make use of Figure 1.) □

3 Solving quadratic equations

Up to now we have dealt with *linear equations* and their roots, that is, equations in which the independent variable appears only as the first power. Let us now advance to *quadratic equations*, which involve the square of the independent variable. The general form of a **quadratic equation** is

$$ax^2 + bx + c = 0 \qquad (15)$$

where a, b and c are constants. There is a simple general method of solving such equations which will be explained shortly, but it is instructive to examine the problem of solving quadratic equations from a number of other viewpoints, since it sheds a good deal of light on the general problem of solving equations.

3.1 The solutions as zeros of a quadratic function

A **quadratic function** is any function $f(x)$ that may be written in the form

$$f(x) = ax^2 + bx + c \qquad (16)(\text{☞})$$

where a, b and c are constants. Thus, when any given quadratic equation is written in the style of Equation 15, its left-hand side will define a corresponding quadratic function $f(x)$. Moreover, it follows from Equations 15 and 16 that $f(x) = 0$ for each value of x that satisfies the given quadratic equation. For this reason the roots (or solutions) of a quadratic equation are also referred to as the **zeros** of the corresponding quadratic function.

If we set $y = f(x)$ and plot the graph of a quadratic function the resulting curve is called a **parabola**. Six different parabolas (corresponding to different values for a, b and c) are shown in Figure 2. If $a > 0$ the parabolas are as shown on the left of the figure, while the case $a < 0$ corresponds to those on the right.

As you can see in Figure 2, the graph of any given quadratic function may meet the x-axis once or twice or not at all — it all depends on the values of a, b and c. However, if the graph does meet the x-axis it does so when $y = f(x) = 0$. It follows that each value of x at which the graph meets the x-axis is a zero of the quadratic function *and* therefore a solution (root) of the corresponding quadratic equation. The possibilities illustrated by the three parts of Figure 2 are as follows:

(a) there are two unequal roots, and the parabola crosses the x-axis in two distinct points;

(b) there is one root, at the point where the parabola just touches the x-axis;

(c) there are no roots, because the parabola never reaches the x-axis.

quadratic equation
quadratic function
zeros (of a function)
parabola

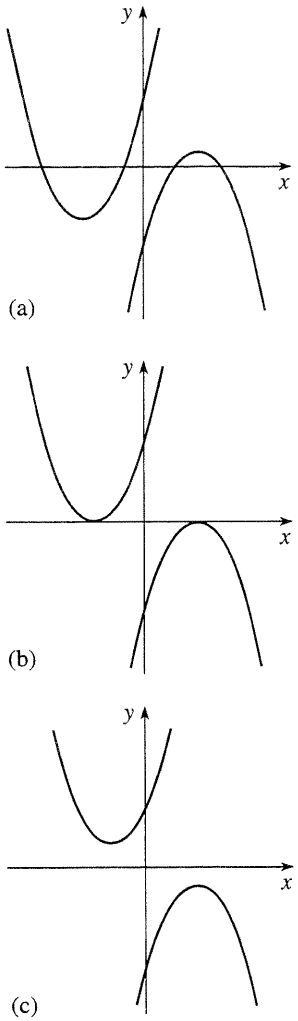


Figure 2 Graphs of six different quadratic functions.

The right-hand side of Equation 16 is sometimes referred to as a *quadratic expression* or simply a *quadratic*.

Note the terminology; equations have *roots* (i.e. solutions) while functions have *zeros*.

The case $a > 0$ gives a 'happy' curve, and $a < 0$ gives a 'sad' curve. ('Happy' and 'sad' are not recognized mathematical terms.)

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Case (b), where there is just one root, is really a special limiting case of (a). If we move either of the parabolas in Figure 2a away from the x-axis its points of intersection with the x-axis (and hence the roots of the corresponding quadratic equation) will get closer and closer together, so that when the parabola just touches the x-axis, the two will have merged. For this reason we usually say that the root in (b) is a **repeated root** or that the equation has **coincident roots**.

repeated root
coincident roots
factorization

These possibilities are the central feature of the process of solving quadratic equations, as we shall see.

3.2 Sums and products of roots: factorizing the quadratic

Throughout this section our aim is to solve the quadratic equation $ax^2 + bx + c = 0$, by finding values for its roots, α and β say. But suppose for the moment that we know α and β and we ask instead 'what is the quadratic equation that has α and β as its roots?'

An answer to this question is given by the equation

$$(x - \alpha)(x - \beta) = 0 \qquad (17)$$

For example, a quadratic equation with roots 2 and –5, is

$$(x - 2)(x + 5) = 0$$

which expands to $x^2 + 3x - 10 = 0$. Any constant multiple of this equation would also have roots 2 and –5.

It is easy to see that if we put either $x = \alpha$ or $x = \beta$ in this equation then the left-hand side is zero (if α and β are equal, then we have a repeated root). Moreover, expanding the left-hand side of Equation 17 gives the quadratic equation

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad (18)$$

In other words, the quadratic equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has α and β as its roots.

Now, compare Equation 18 with the general quadratic equation we are trying to solve:

$$ax^2 + bx + c = 0 \qquad \text{(Eqn 15)}$$

We can make the general equation look more like Equation 18 by dividing both sides by a, which gives

$$x^2 + (b/a)x + (c/a) = 0 \qquad (19)$$

It then becomes clear that Equations 18 and 19 are the same, provided that

$$\alpha + \beta = -b/a \quad \text{and} \quad \alpha\beta = c/a \qquad (20)$$

Thus, we can make the following connections between the coefficients of a quadratic and its roots.

> Given a general quadratic equation $ax^2 + bx + c = 0$ with roots α and β
> - the sum of its roots is $\alpha + \beta = -b/a$
> - and the product of its roots is $\alpha\beta = c/a$

Unfortunately it is not possible to use these equations directly to solve the equation; if you try to do so you will just end up with the original equation! However, they may help us to *factorize* the original equation, that is to re-express it in the form of Equation 17 and hence identify its solutions. This is often the quickest method of solution, especially if we know, or suspect, that the roots are integers (i.e. whole numbers).

◆ Solve the equation $x^2 - 5x + 6 = 0$.

◇ In this case $a = 1$, $b = -5$ and $c = 6$, so we wish to find roots, α and β, such that $\alpha + \beta = 5$ and $\alpha\beta = 6$. If the roots are integers then, apart from signs, they must either be 2 and 3, or 1 and 6, and both members of the pair must have the same sign, since their product is +6. In this case only the values 2 and 3 meet all the requirements. This means that the equation $x^2 - 5x + 6 = 0$ can be written in the alternative form $(x - 2)(x - 3) = 0$, and *now* we can see that the roots are 2 and 3. □

The fact that there are very few choices of integers that multiply to give 6 helps us considerably in this case.

In the above solution we have used the process known as **factorization** to find the roots of the equation. We can check that our answer is correct by expanding the brackets to give

$$(x - 2)(x - 3) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$$

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If we had to solve the almost identical equation

$$x^2 - 5x - 6 = 0$$

we would have $\alpha + \beta = 5$ and $\alpha\beta = -6$ and the roots would still involve 2 and 3, or 1 and 6, but this time the two numbers in each pair would have opposite signs, since the product $\alpha\beta$ is negative. We must choose the pair + 6 and −1. We can check by expanding as before, and this time:

$$(x - 6)(x + 1) = x^2 - 6x + x - 6 = x^2 - 5x - 6$$

Question T10 Solve the following quadratic equations by factorization:

(a) $x^2 - 3x + 2 = 0$ (b) $x^2 + x - 2 = 0$ (c) $x^2 + 7x + 12 = 0$ (d) $x^2 - x - 12 = 0$

(e) $x^2 - 2x - 15 = 0$ (f) $x^2 - 4x - 12 = 0$ □

The above equations all had the coefficient of x^2 equal to 1. It is a little more complicated if a is not 1; consider, for example, the expression

$$2x^2 + 5x + 3$$

If this can be factorized, then it must be possible to express it as

$$(2x + ?)(x + ?)$$

Trial and error shows us that the factorization is

$$(2x + 3)(x + 1)$$

Thus, to solve the equation $2x^2 + 5x + 3 = 0$ we rewrite it as $(2x + 3)(x + 1) = 0$ and then it is clear that the roots are −3/2 and −1. (☞)

If $(2x + 3)(x + 1) = 0$ then *either* $(2x + 3) = 0$ or $(x + 1) = 0$. Thus the roots are $x = -3/2$ and $x = -1$.

It is important to check your answer by substituting the values you have obtained back into the original equation to verify that they are indeed roots of the equation.

Question T11 Solve the following equations by factorization:

(a) $2x^2 + 9x - 5 = 0$ (b) $3x^2 - 11x - 4 = 0$ (c) $3x^2 - x - 2 = 0$

(d) $4x^2 - 3x - 1 = 0$ (e) $4x^2 - 4x + 1 = 0$ (f) $6x^2 + 5x - 6 = 0$ □

With practice this method can become very quick, but unfortunately it does not always work. The quadratic equation

$$x^2 - 3x + 1 = 0$$

cannot be factorized in the same way, because the roots are not integers. We must therefore find a more generally applicable method even if it is somewhat slower.

3.3 The quadratic equation formula: completing the square

The quadratic equation

$$x^2 - 7 = 0$$

is easily solved, for we may simply add 7 to both sides of the equation and then take square roots of both sides to obtain

$$x^2 = 7, \text{ so that } x = \pm\sqrt{7}$$

In *FLAP* square roots are generally regarded as representing quantities that may be positive or negative, so strictly speaking the ± sign is superfluous. However, we are retaining it here and treating it as a *reminder* that a square root may be positive or negative.

The quadratic equation

$$(x + 3)^2 - 5 = 0$$

is hardly more difficult, since we can add 5 to both sides, then take the square root of both sides to get $x + 3 = \pm\sqrt{5}$ so that $x = -3 \pm \sqrt{5}$.

If we could reduce the general quadratic equation to something like this form we should have another method of solution that might work more generally. Introducing such a method is the main purpose of this subsection, but first we must examine how one might write *any* quadratic equation in such a form. Another example may make the problem clear.

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perfect square
completing the square

To solve the equation

$$3x^2 - 10x + 1 = 0$$

we first extract the factor 3 to give

$$3\left(x^2 - \frac{10}{3}x + \frac{1}{3}\right) = 0 \qquad (21)$$

and we thus ensure that the coefficient of x^2 inside the brackets is 1. Next we arrange for the term $x^2 - \frac{10}{3}x$ to arise from a term of the form $(x + k)^2$ for some value k. (Such a term is called a **perfect square**.) Since $(x + k)^2 = x^2 + 2kx + k^2$ we will be forced to choose k so that $2k = -10/3$ in this case, in other words $k = -5/3$, which gives $k^2 = 25/9$. Now we add and subtract 25/9 to the term in the brackets in Equation 21 to get

$$3\left(x^2 - \frac{10}{3}x + \frac{25}{9} - \frac{25}{9} + \frac{1}{3}\right) = 0 \qquad (22)$$

The advantage of this step is that the first three terms in the bracket can be written as a perfect square, so Equation 22 becomes

$$3\left[\left(x - \frac{5}{3}\right)^2 - \frac{25}{9} + \frac{1}{3}\right] = 0$$

and this can be simplified to

$$3\left[\left(x - \frac{5}{3}\right)^2 - \frac{22}{9}\right] = 0$$

so that $\left(x - \frac{5}{3}\right)^2 = \frac{22}{9}$ and therefore $x = \frac{5}{3} \pm \frac{\sqrt{22}}{3}$

The above process is known as **completing the square**; try the same method in the following exercises.

Question T12 Find the roots of the following equations by completing the square:
(a) $x^2 - x - 2 = 0$ (b) $x^2 - 2x - 8 = 0$ (c) $2x^2 + x - 10 = 0$
(d) $3x^2 - 18x + 24 = 0$ □

To see how this method can be applied in general, let us repeat the procedure for the general quadratic.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

Now if we put this expression equal to zero, we can divide both sides by the multiplying factor a, and get

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

so that $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

and $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

thus $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (23)$$

This is a very important formula, which is used so often that you should commit it to memory. (The author was taught the piece of doggerel in the margin; you may, or



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may not, find it helpful.) In practice this formula provides the simplest and quickest way of solving most quadratic equations.

discriminant

Question T13 Use Equation 23 to solve the problems in Question T12. □

Question T14 Use Equation 23 to solve the following:

(a) $x^2 - 6x + 1 = 0$ (b) $3x^2 - 9x + 2 = 0$

Do not use your calculator, but leave the answer in the simplest form possible, with the square roots not evaluated. Check that the sum and product of the roots are correct. □

Not only does Equation 23 simplify the process of solving quadratic equations, it also has a feature which enables us to say something about the general properties of the quadratic equation without actually having to solve it. This feature is known as the **discriminant**, and is given by the expression inside the square root, $b^2 - 4ac$.

The sign of the discriminant determines some useful properties of the roots of the equation:

- If $b^2 - 4ac$ is positive, then we can find its square root, and the quadratic equation will have two different roots. This corresponds to the case (a) of Figure 2, where the parabola crosses the x-axis twice. (For reasons that will become clear in the next subsection, we also say that the equation has two distinct *real* roots.)
- If $b^2 - 4ac$ is equal to zero, then the two roots of the equation will be repeated and the quadratic equation can be written as $a(x + k)^2 = 0$. This corresponds to the special case (b) of Figure 2 in which the parabola just touches the x-axis and we have just one (repeated) real root.
- If $b^2 - 4ac$ is negative, then we cannot take the square root, and we say that there are no real roots to the equation. This case corresponds to case (c) of Figure 2, when the parabola does not intersect the x-axis at all.

Question T15 Use the discriminant to establish how many real roots the following equations have:

(a) $4x^2 - 20x + 25 = 0$ (b) $x^2 + x + 6 = 0$ (c) $x^2 + x - 6 = 0$ □

Although we now seem to have a recipe that will solve all our problems with quadratic equations, and make the whole business purely mechanical, this is not quite true. It often happens that when we have to substitute numbers in formulae, things are not quite as straightforward as appears at first sight. For example, consider the equation:

$$x^2 - 2x + 10^{-12} = 0$$

and notice that the constant term, c, is very small.

If the constant term 10^{-12} were not there, the equation would be $x^2 - 2x = 0$ which has roots at $x = 2$ and $x = 0$, so we would expect the given equation to have roots near these values.

Applying the formula (Equation 23) gives us the roots

$$x = \left(2 \pm \sqrt{4 - 4 \times 10^{-12}}\right)/2$$

which can only be evaluated as 2 or 0 to the accuracy of most calculators.

This shows the value of remembering the relations between the roots and coefficients of a quadratic equation as well as the formula.

The first root will certainly be as close to 2 as most people could wish, but to get the second root we have to take the difference of two numbers that are very close together, which means that we have lost a lot of precision. However, there is an alternative. If we calculate the larger root (say α) using Equation 23, we find a value near to 2 as before. We can then use the fact that the product of the roots ($\alpha\beta$) is equal to c/a, i.e. 10^{-12}, to obtain the value $10^{-12}/2 = 5 \times 10^{-13}$ for the smaller root (say β).

3.4 Roots for all equations: complex numbers

Let us look again at what happens when the discriminant $b^2 - 4ac$ is negative. Consider the equation

$$x^2 - x + 1 = 0$$

for which $a = 1$, $b = -1$ and $c = 1$, so that $b^2 - 4ac = -3$.

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If we try to solve this equation using Equation 23 we get

$$x = \frac{1}{2}\left(1 \pm \sqrt{-3}\right)$$

which does not seem to make sense, since a negative number like -3 has no square root; yet if we substitute one of these 'roots' into the left-hand side of the equation we find:

$$\begin{aligned} x^2 - x + 1 &= \left[\frac{1}{2}\left(1 - \sqrt{-3}\right)\right]^2 - \frac{1}{2}\left(1 - \sqrt{-3}\right) + 1 \\ &= \frac{1}{4}\left[1 - 2\sqrt{-3} + \left(\sqrt{-3}\right)^2\right] - \frac{1}{2}\left(1 - \sqrt{-3}\right) + 1 \\ &= \frac{1}{4}\left(1 - 2\sqrt{-3} - 3\right) - \frac{1}{2}\left(1 - \sqrt{-3}\right) + 1 \\ &= -\frac{1}{2} - \frac{1}{2}\sqrt{-3} - \frac{1}{2}\left(1 - \sqrt{-3}\right) + 1 = 0 \end{aligned}$$

So treating the quantity $\sqrt{-3}$ as an 'ordinary' number such that its square is -3 appears to give us a solution to the equation.

Question T16 Show that

$$x = \frac{1}{2}\left(1 + \sqrt{-3}\right)$$

also satisfies the equation in the same way. □

Both 'roots' satisfy the equation, even if they do not mean anything! Since there seems to be no contradiction, this is an invitation to mathematicians to invent something new. A quantity called i is introduced, which is used in place of $\sqrt{-1}$; it has the property that $i \times i = -1$. This enables us to deal with the square root of any negative number; for example $\sqrt{-3} = \sqrt{-1 \times 3} = \sqrt{-1} \times \sqrt{3} = i\sqrt{3}$. The number i can be given a perfectly sound mathematical definition, but that is not our concern here, we merely wish to use it. For example, we may write

Some engineers, and physicists who work with complex numbers use j instead of i, but all mathematicians, and most physicists, use i.

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

or $$i^3 = i \times i \times i = -1 \times i = -i$$

or $$\frac{1}{i} = \frac{i}{i \times i} = \frac{i}{-1} = -i$$

The quantity i and all its multiples, such as $i\sqrt{3}$, are examples of **imaginary numbers**. These should be contrasted with 'ordinary numbers' such as 35 or -2.6 that do not involve i and which are formally referred to as **real numbers**. By adding together real and imaginary numbers it is possible to create so-called **complex numbers** of the form

$$a + bi$$

where a and b are both real numbers. For example, $3 + 5i$, would be a typical complex number.

The real number a is called the **real part**, and the real number b the **imaginary part**, of the complex number. The real part of $3 + 5i$ is 3 while its imaginary part is 5. 'Real' and 'imaginary' are perhaps unfortunate names to have been chosen, since they suggest that a is more important than b, whereas we must treat both on an equal footing. However, the names are now too well established to be changed easily.

Notice that the imaginary part of $3 + 5i$ is 5 *not* $5i$.

Question T17 Using complex numbers, find the solutions of the following equations:

(a) $x^2 + 25 = 0$ (b) $x^2 + x + 6 = 0$ □

Complex numbers play an important part in mathematics and physics, and are the subject of a whole block of *FLAP* modules. We cannot go into all their applications here, but it is worth noting that if we allow roots to be complex, and if we count repeated roots twice, then *every* quadratic equation has *two* roots, the values of which are given by Equation 23.

In physics and engineering, complex numbers are invaluable in calculations involving electrical circuits, where the two parts can be associated with the oscillatory and decay properties of the circuits. They are also useful in the study of waves and optics, and in many other areas.

imaginary numbers
real numbers
complex numbers
real part
imaginary part

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4 Solving more complicated equations

cubic equations
quartic equations
polynomial equations
degree
polynomial expression
polynomial function
fundamental theorem of algebra
algebraic division

The equations we have dealt with so far are a very small sample of the variety that you will meet. We mentioned simultaneous equations with more than two unknown quantities in Subsection 2.3, but you will also meet equations that involve higher powers of the unknown variable than the square, including **cubic equations** such as

$$x^3 + x^2 + x + 1 = 0$$

and **quartic equations** such as

$$x^4 + x^3 + x^2 + x + 1 = 0$$

Such equations, together with the linear and quadratic equations we considered earlier, are special cases of a class of **polynomial equations** which have the general form

$$a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n = 0$$

where n is an integer called the **degree** of the polynomial, and the $n + 1$ constants $a_0, a_1, a_2, \ldots a_{n-1}$ and a_n are the *coefficients* of the polynomial. The left-hand side of such an equation constitutes a **polynomial expression** and may be used to define a corresponding **polynomial function**.

There is a general result for roots of polynomial equations:

> The **fundamental theorem of algebra** states that *every polynomial equation of degree n has precisely n roots* as long as complex roots and repeated roots are counted.

Cubic and quartic equations can be solved using formulae similar to that used for solving quadratic equations, but these formulae are so complicated that they are rarely used. However, there are no general formulae for the roots of equations of the 5th and higher degrees, and in such cases we must usually be satisfied with numerical approximations. Many of the numerical methods that are used for finding these approximations can also be applied to non-polynomial equations and we shall examine such methods shortly. First, however, let us look at methods that allow us to solve *some* polynomial equations.

4.1 Solving polynomial equations

Sometimes, if we are fortunate, we are able to guess one, or more, solutions of a cubic, or higher order, equation and this makes the task of finding the remaining roots considerably easier. For example, if we substitute $x = 1$ into the left-hand side of the following equation:

$$x^3 - 6x^2 + 11x - 6 = 0 \qquad (24)$$

we see immediately that this value of x makes the equation true. It follows that $(x - 1)$ is a factor of the left-hand side of Equation 24 so that

$$x^3 - 6x^2 + 11x - 6 = (x - 1) \times (\text{a quadratic expression in } x)$$

$$\text{i.e. } \left(\frac{x^3 - 6x + 11x - 6}{x - 1}\right) = \text{a quadratic expression in } x$$

We can find this quadratic expression by **algebraic division** as follows:

$$\begin{array}{r} x^2 - 5x + 6 \\ x - 1 \overline{)\, x^3 - 6x^2 + 11x - 6} \\ \underline{x^3 - x^2} \\ -5x^2 + 11x \\ \underline{-5x^2 + 5x} \\ +6x - 6 \\ \underline{+6x - 6} \\ \cdots\cdots \end{array} \quad (\text{☞})$$

so that $x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$.

Note the general form

$$\text{denominator} \overline{)\, \text{numerator}}^{\text{result}}$$

The result on the top is worked out term by term starting with x^2. We multiply the denominator by x^2 to get $x^3 - x^2$. This is written below the numerator and subtracted from it, leaving $-5x^2 + 11x - 6$. We then proceed to divide this by the dominator $x - 1$, thus finding that the second term in the result is $-5x$, and so on.

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The quadratic expression on the right is then easily factorized to give

$$x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$$

and now we can see that the roots of Equation 24 are 1, 2 and 3.

This method is quite restricted in its applications because it relies on our ability to guess a root, but it can still be very useful. (☞)

Some computer programs that perform algebraic manipulations will factorize expressions of this kind.

Question T18 Find the roots of $x^3 + x^2 - 4x - 4 = 0$. (Hint: try some integer values of x.) □

4.2 Graphical solutions

In Subsection 2.4 we saw how we could use graphs to solve a pair of simultaneous equations, and in Section 3 we saw how a graph of a quadratic function could give us clues about the roots of the corresponding quadratic equation. Similar methods can be applied to a wide variety of equations. Consider, for example, the function

$$f(x) = x^3 - 4x + 1 \tag{25}$$

the graph of which is shown in Figure 3.

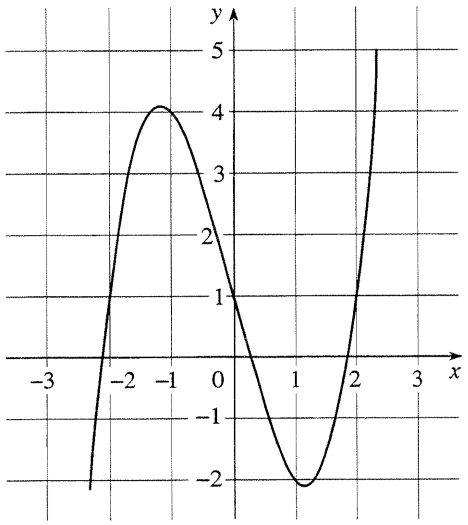


Figure 3 The graph of Equation 25.

The curve crosses the x-axis three times, so the equation $f(x) = 0$ has three roots, which are, respectively, near -2, between 0 and 1, and near $+2$. A larger scale graph would tell us that the values are nearer -2.1, 0.25 and $+1.9$. If this degree of precision is enough, then nothing further need be done. If greater accuracy is needed, then one has the choice of drawing a yet more accurate graph (or perhaps three separate graphs, one for the neighbourhood of each root) or of using another method. Since more function values must be calculated to plot the new graphs, it is probably better to go straight to one of the *iterative* methods discussed in the next subsection.

Any pair of simultaneous equations can be solved approximately by graphical methods. For example, the equations

$$y = x^3 - 4x + 1 \tag{26}$$

and

$$4y = 3 - 2x - x^2 \tag{27}$$

can be solved by plotting the graphs of $y = x^3 - 4x + 1$ and $y = (3 - 2x - x^2)/4$. The points of intersection correspond to the solutions of this pair of equations. Figure 4 shows that there are three possible solutions, corresponding to the three intersections, at approximately

$$x = -2, \quad y = 0.7$$

$$x = 0, \quad y = 0.7$$

and

$$x = 1.7, \quad y = -0.8$$

which can be abbreviated to

$$(-2, 0.7), (0, 0.7), (1.7, -0.8)$$

More accurately, the points of intersection are

$x = -2.0325$ $y = 0.7336$

$x = 0.0719$ $y = 0.7128$

$x = 1.7106$ $y = -0.8369$

(but these values were found by numerical, rather than graphical, methods).

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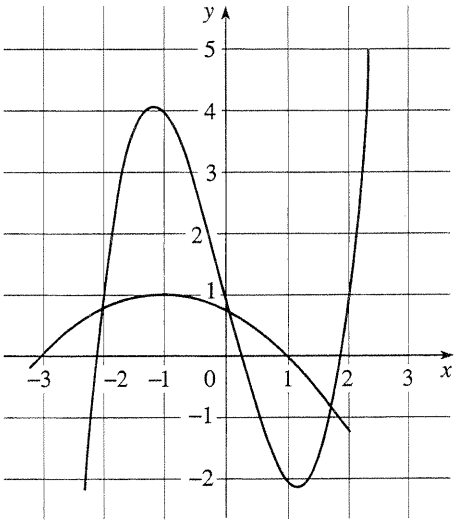


Figure 4 The graphs of Equations 26 and 27.

numerical procedures
bisection method
regula falsi

4.3 Numerical procedures

Numerical procedures are methods of finding approximate solutions to a particular equation, in which all the constants are known numbers.

The bisection method

The simplest procedure, called the **bisection method**, is not much more than trial and error. If we want to solve the equation $f(x) = 0$, and we have some idea of the solution, perhaps from a graph, then we evaluate $f(x)$ on either side of our initial estimate so as to obtain a range of x values over which the sign of $f(x)$ changes. As an illustration, consider the equation

$$x^3 - 4x + 1 = 0$$

corresponding to the function $f(x) = x^3 - 4x + 1$. From Figure 3 we see that there is a root between $x = 1$ and $x = 2$, and this is confirmed by noticing that

$$f(1) = -2 \quad \text{and} \quad f(2) = 1$$

so that there is a sign change on the interval from $x = 1$ to $x = 2$.

At the mid-point, $x = 1.5$, we have $f(1.5) = -1.625$ so that the sign change (and therefore the root) must lie between $x = 1.5$ and $x = 2$.

Similarly, we can show that the root lies between $x = 1.75$ and $x = 2$; and we can continue in this fashion, bisecting the interval and evaluating the function at the mid-point, as far as we please and so obtain an approximation to the root which is as accurate as we need.

Regula falsi

The method of bisection is very reliable and simple to implement as a computer program. It is true that it is comparatively slow, but with the speed of modern computers this hardly matters for a single calculation. However, speed may become a critical factor if you need to perform a large number of evaluations of roots. A simple method of speeding up the method considerably is the variation known as ***regula falsi***. Let us look in detail at a graph of the neighbourhood of the solution, in Figure 5.

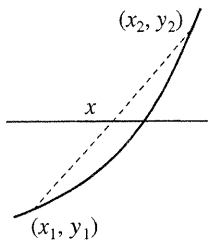


Figure 5 *Regula falsi.*

If x_1 and x_2 are initial estimates for the root, and if $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have opposite signs, then a straight line between the two points will intersect the x-axis at a point which is closer to the root. The equation of the straight line joining the two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2} \tag{28}$$

This is a version of the *two-point* form of the equation of a straight line. See the *Glossary* for details.

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If we put $y = 0$ in this equation, then the corresponding value of x will determine the point at which the line crosses the horizontal axis and will therefore give an improved estimate of the root. Thus,

$$\frac{-y_1}{x - x_1} = \frac{-y_2}{x - x_2}$$

which can be rearranged to give

$$x = \frac{x_2 y_1 - x_1 y_2}{y_1 - y_2}$$

This improved estimate can be used as the starting point for an even better estimate.

Finding a root by this method takes fewer steps, although more work is required in each step. In practice it can be a fine judgement as to which method is more efficient in a particular case.

Question T19 The function defined in Equation 25 has a zero near $x = 2$. Find this zero to an accuracy of five decimal places (i.e. so that the equation is satisfied to five decimal places) using one of the two numerical methods above. You may use your calculator. □

Iterative methods

iterative methods
iteration formula
iteration

An alternative class of methods that are often useful are the **iterative methods**; these are based on the idea that if one has an approximate solution to an equation, then it can be used to get a better value. You might consider either of the previous methods to be in this class, but we usually mean methods in which the next approximation is obtained by a special formula. Look again at the equation $x^3 - 4x + 1 = 0$, which we know, from Figure 3, has a solution near $x = 2$. We can rewrite the equation as

$$x = \frac{2x^3 - 1}{3x^2 - 4} \qquad (29)(\text{☞})$$

You may wish to verify that

$$x = \frac{2x^3 - 1}{3x^2 - 4}$$

is a rearrangement of the equation

$$x^3 - 4x + 1 = 0$$

Don't worry about how we found this rearrangement — that is dealt with in another module under the heading *Newton–Raphson formula* (see the *Glossary*).

and we can use this to provide an **iteration formula** that relates an approximate root x_n to a more accurate approximate root x_{n+1}

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 4} \qquad (30)$$

So, if we set $n = 1$ in Equation 30 and take as our crude first approximation $x_1 = 2$, we can use Equation 30 to find a better approximation $x_2 = 15/8 = 1.875$. Setting $n = 2$ in Equation 30 we can now repeat this process (starting with $x_2 = 1.875$) to find an even better approximation x_3. Continuing in this way we can obtain a succession of approximations x_2, x_3, x_4, x_5 and so on, with each step, or **iteration**, being an improvement on the last.

$x_2 = 1.875$
$x_3 = 1.860\,978\,52$
$x_4 = 1.860\,805\,88$
$x_5 = 1.860\,805\,85$
$x_6 = 1.860\,805\,85$

Notice that after three iterations the solution is consistent with the answer to Question T19 which was given to five decimal places.

We stop when two successive iterations are the same to the required number of decimal places, since we can expect no further improvement.

Question T20 The equation $x^3 + 2x - 1 = 0$, which has a root near $x = 0.45$, can be rearranged to give

$$x = \frac{1 - x^3}{2}$$

Using the first approximation $x_1 = 0.450\,00$ and the iteration formula

$$x_{n+1} = \frac{1 - x_n^3}{2}$$

find this root to five decimal place accuracy. (You may use your calculator.) □

Unfortunately, this simple method must be used with great care, and careful choice of the iteration formula is often needed to produce an iteration that will result in a meaningful answer. However, if the method does work, then it has the great

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advantage that it is almost proof against a mistake, since any reasonable approximation will be improved; the worst that can happen is that the process is slowed down somewhat.

More advanced methods, beyond the scope of this module, enable one to produce very rapid convergence to the root.

Question T21 The following iterative formula was used in the early days of computers to find the *reciprocal* of a given number a

$$x_{n+1} = x_n(2 - ax_n)$$

Use this iteration to find 1/0.8 (i.e. let $a = 0.8$ in the above formula), starting with $x_1 = 1$. (You may use your calculator.) □

5 Closing items

5.1 Module summary

1 Given a function $f(x)$, a *solution* or *root* of the equation $f(x) = 0$ is a value of x that makes the equation true. Such a value is also a *zero* of the function and is represented graphically by any value of x at which the graph of $f(x)$ meets the x-axis.

2 A *linear equation* in a single variable x has the general form $ax + b = 0$ and may be *solved* by rearrangement to give $x = -b/a$.

3 A pair of *simultaneous linear equations* may be solved by *substitution*; the method of *elimination* provides a standard routine for carrying out the substitution.

4 A set of simultaneous linear equations involving n unknowns may be solved, provided that it contains at least n *independent* equations and provided that none of the equations are *inconsistent*.

5 Simultaneous linear equations may also be solved *graphically*, by drawing appropriate straight lines and determining their point of intersection.

6 A *quadratic equation* $ax^2 + bx + c = 0$ has roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, as may be demonstrated by the process of *completing the square*.

7 If the *discriminant*, $b^2 - 4ac$, of a quadratic equation is positive there are two *real* roots, if it is zero there is a *repeated* root, and if the discriminant is negative there are no real roots. In the latter case it is possible to find two *complex* roots of the form $a + bi$ where $i = \sqrt{-1}$.

8 If the roots of $ax^2 + bx + c = 0$ are α and β then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$. Such a quadratic may be *factorized* and written in the form $(x - \alpha)(x - \beta) = 0$.

9 According to the *fundamental theorem of algebra*, a *polynomial equation* of *degree n* has n roots, provided we count complex roots and repeated roots.

10 Equations in general can be tackled by *graphical*, *numerical* and *iterative* techniques, but each case must be treated on its merits.

11 When solving equations that involve physical quantities, don't introduce numbers until it is essential; leave all the arithmetic until the end.

12 Always check that your answers satisfy the original equations and that they make sense.

5.2 Achievements

After completing this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Solve simple linear equations by rearrangement.

A3 Recognize an equation that is linear in a function of the unknown, and rearrange it for solution.

A4 Solve linear simultaneous equations by substitution, provided they are consistent and independent.

A5 Use graphical methods to solve simultaneous linear equations.

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A6 Recall the general form of a quadratic equation and sketch the graph of any given quadratic function.

A7 Relate the coefficients of a given quadratic equation to the sum and product of the roots of that equation.

A8 Factorize a simple quadratic function, and so solve the corresponding quadratic equation.

A9 Explain the method of completing the square, and use it to solve a quadratic equation.

A10 Write down the general formula for the roots of a quadratic equation, and use it to solve any particular equation.

A11 Recall the definition of the discriminant, and explain how it determines the nature of the solutions of a quadratic equation.

A12 Recall the nature and significance of complex numbers, and use them in the solution of quadratic equations.

A13 Solve simple polynomial equations by factorization.

A14 Describe the use of graphical methods in the solution of general equations, and use them to locate the roots of various equations.

A15 Use the methods of bisection and *regula falsi* in the solution of simple equations.

A16 Use iterative methods in the solution of equations given the iteration formula.

5.3 Exit test

Study comment Having completed this module, you should be able to answer the following questions, each of which tests one or more of the *Achievements*.

Question E1 (*A2*) Find x if $2(x-1)+3(x-6)=0$.

Question E2 (*A3*) Find x if

$$\frac{1}{x-2}+2\left(\frac{1}{x-2}-6\right)=0$$

Question E3 (*A4*) Solve the following simultaneous equations:

$$2x+y=-4 \qquad 3x+5y=1$$

Question E4 (*A4* and *A5*) Solve the following simultaneous equations first by drawing suitable graphs, then algebraically

$$x+y=k \qquad 2x-y=1$$

for $k = 2$, 5 and 8.

Question E5 (*A6* and *A11*) Sketch graphs to show quadratic functions with (a) two distinct zeros, (b) two coincident zeros, and (c) no real zeros.

Question E6 (*A7* and *A8*) Solve the following equations by factorizing the left-hand sides:

(a) $x^2+5x-14=0$ (b) $2x^2-21x+27=0$

Check your answers by showing that the sum and product of the roots are correct in each case.

Question E7 (*A9* and/or *A10*) Solve the following equations:

(a) $x^2-2x-1=0$ (b) $3x^2+x-2=0$

Question E8 (*A11*) State the conditions for the equation (in the variable x)

$$x^2-2rx+s=0$$

to have (a) two real roots, and (b) repeated roots.

Question E9 (*A10*, *A11* and *A12*) Write down the roots of the equation in Question E8 above if neither of the conditions quoted there is fulfilled (so that the equation has complex roots).

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Question E10 (*A13*) Guess one of the roots, and then solve the following equation by factorization:

$$x^3 - 4x^2 + x + 6 = 0.$$

Question E11 (*A14*) Sketch the graph of the function

$$g(x) = 3x^3 - 11x^2 + 5x + 8$$

and give approximate values for its zeros. (You may use your calculator.)

Question E12 (*A15*) Use a numerical method (and the results of Question E11 above) to find the largest root of the equation $3x^3 - 11x^2 + 5x + 8 = 0$ to two decimal places. (You may use your calculator.)

Study comment Now that you have completed the *Exit test*, go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

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1.1

6 Answers and comments

Fast track answers

F1 $x = -3$.

F2 $x = 2, y = 1$.

F3 $x = \frac{1}{2}(-1 \pm \sqrt{5})$. The sum of the roots is -1, and their product is also -1. (If you do not understand the previous sentence you should take this as an indication that you should read the module.)

Ready to study answers

R1 (a) $a^2 - 4b^2$, (b) $x^2 - x - 12$, (c) $p^2 - 6p + 9$.

R2 $12x$.

R3 $f(2y) = 3(2y) + 2 = 6y + 2$;
$f(y - 1) = 3(y - 1) + 2 = 3y - 1$.

R4 $+x^2$ and $-x^2$.

R5 The graphs of the two functions are given in Figure 6.

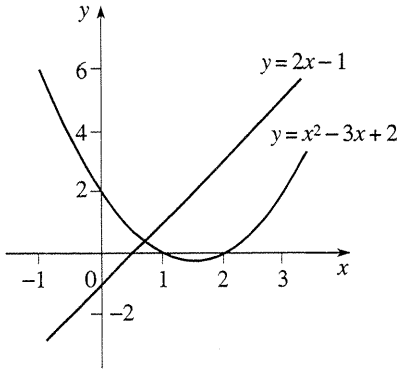


Figure 6 See Answer R5.

R6 $\dfrac{1}{1-x} - \dfrac{1}{1+x} = \dfrac{(1+x)-(1-x)}{(1-x)(1+x)} = \dfrac{2x}{1-x^2}$

Text answers

T1 $2(x + 1) = x - 5$; this is equivalent to $2x + 2 = x - 5$, therefore $2x - x = -5 - 2$ and $x = -7$.

To check this solution, substitute the value for x in the equation: left-hand side $= 2(-7 + 1) = -12$; right-hand side $= -7 - 5 = -12$.

T2 $z - 2 + 6z + 3 - 5z + 7 = 0$; if the terms are collected, $z + 6z - 5z - 2 + 3 + 7 = 0$; $2z + 8 = 0$; so $z = -4$.

To check this solution, substitute the value for x in the equation: $-6 + 3(-7) - (-20 - 7) = -6 - 21 + 27 = 0$.

T3 In terms of a and b the solution for p is $p = -\frac{1}{2}(a + 2b)$; now if we substitute for a and b, we obtain $p = -\frac{1}{2}(1 + 8) = -9/2$.

T4 Notice that we do not need to solve the equation again. We simply substitute the new values for a and b in the equation found in Answer T3, therefore, $p = \frac{1}{2}(4 - 4) = 0$.

T5 (a) $\dfrac{1}{x-1} = 6$, so $1 = 6(x - 1)$ or $6x = 7$ and therefore $x = 7/6$.

(b) There is no cancellation here; multiplying both sides by $(x + 1)$ gives $(x + 1)(x^2 - 1) = 2$ which expands to give $x^3 + x^2 - x - 1 = 2$, so this equation cannot be reduced to linear form.

(c) If we expand the bracket, $x^2 + 2 = x^2 - 3x$, so $2 = -3x$ or $x = -2/3$.

(d) If we expand the left-hand side we get $\dfrac{x^2 - 4}{x^2 - 4} = x$ which gives (after cancellation) $x = 1$.

T6 If we put

$$y = \frac{x^2 + 1}{x^2 - 1}$$

then $2(y + 1) = y + 5$; $2y + 2 = y + 5$; so $y = 3$.

If we then substitute for y, we have

$$\frac{x^2 + 1}{x^2 - 1} = 3;\ x^2 + 1 = 3(x^2 - 1);\ 2x^2 = 4;\ x^2 = 2;$$

so $x = \pm\sqrt{2}$. If we put $x = +\sqrt{2}$ in the original equation to check this, we find the left-hand side $= 2\left(\dfrac{2+1}{2-1} + 1\right) = 8$ and the right-hand side $= \dfrac{2+1}{2-1} + 5 = 8$. This can then be repeated for $x = -\sqrt{2}$.

T7 (a) Multiply the first equation by 2 and the second by 5:

$10x + 4y = 0$

$10x + 25y = 105$

subtract the first equation from the second:

$21y = 105$

so $y = 5$; substituting this value in either equation gives $x = -2$; you can then use the other one to check the answer.

(b) Multiply the first equation by 2:

$8x + 2y = 20$

$3x - 2y = 13$

this time add the two equations:

$11x = 33$

so $x = 3$; substituting this value in either equation gives $y = -2$. Did you remember to check the solution?

T8 We have $a = 1$, $b = 2$, $c = 2$ and $d = -3$, so that $D = 1 \times (-3) - (2 \times 2) = -7$. Also $p = 8$ and $q = -5$ so that (from Equation 12)

$$x = \frac{[(-3) \times 8] - [2 \times (-5)]}{-7} = 2$$

and (from Equation 13)

$$y = \frac{-5 - (2 \times 8)}{-7} = 3$$

As you can see, the amount of work using this general method is *not* significantly less than the previous methods. Hence our reluctance to use it.

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T9 The intersection for various value of k is shown in Figure 7.

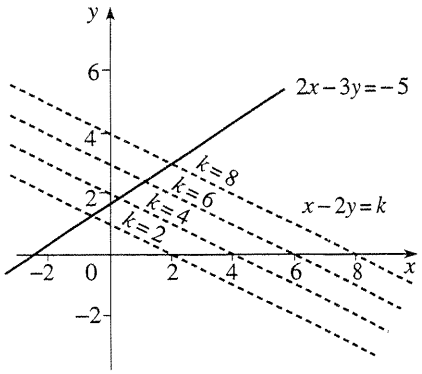


Figure 7 See Answer T9.

$k = 2$; $x \approx -0.6$, $y \approx 1.3$

$k = 4$; $x \approx 0.3$, $y \approx 1.8$

$k = 6$; $x \approx 1.1$, $y \approx 2.4$

$k = 8$; $x \approx 2.0$, $y \approx 3.0$

T10 (a) $(x-1)(x-2) = 0$, so $x = 1$ or $x = 2$.

(b) $(x-1)(x+2) = 0$, so $x = 1$ or $x = -2$.

(c) $(x+3)(x+4) = 0$, so $x = -3$ or $x = -4$.

(d) $(x-4)(x+3) = 0$, so $x = 4$ or $x = -3$.

(e) $(x-5)(x+3) = 0$, so $x = 5$ or $x = -3$.

(f) $(x+2)(x-6) = 0$, so $x = -2$ or $x = 6$.

T11 (a) $(2x-1)(x+5)$, so $x = 1/2$ or $x = -5$.

(b) $(3x+1)(x-4) = 0$, so $x = -1/3$ or $x = 4$.

(c) $(3x+2)(x-1) = 0$, so $x = -2/3$ or $x = 1$.

(d) $(4x+1)(x-1) = 0$, so $x = -1/4$ or $x = 1$.

(e) $(2x-1)(2x-1) = 0$, so $x = 1/2$ (repeated root).

(f) $(3x-2)(2x+3) = 0$, so $x = 2/3$ or $x = -3/2$.

T12 (a) $x^2 - x - 2 = x^2 - x + \frac{1}{4} - \frac{1}{4} - 2 = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$

hence $x - \frac{1}{2} = \pm\sqrt{9/4}$; $x = \frac{1}{2} \pm \frac{3}{2} = 2$ or -1.

(b) $x^2 - 2x - 8 = x^2 - 2x + 1 - 1 - 8 = (x-1)^2 - 9$

hence $x - 1 = \pm\sqrt{9}$; $x = 1 \pm 3 = 4$ or -2.

(c) $2x^2 + x - 10 = 2\left(x^2 + \frac{x}{2} - 5\right)$

$= 2\left(x^2 + \frac{x}{2} + \frac{1}{16} - \frac{1}{16} - 5\right) = 2\left[\left(x + \frac{1}{4}\right)^2 - \frac{81}{16}\right]$

hence $\left(x + \frac{1}{4}\right)^2 = \frac{81}{16}$; $x + \frac{1}{4} = \pm\frac{9}{4}$; $x = +2$ or -2.5.

(d) $3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x^2 - 6x + 9 - 9 + 8)$
$= 3[(x-3)^2 - 1]$.
Hence $(x-3)^2 - 1 = 0$; $x - 3 = \pm 1$; $x = 4$ or 2.

T13 The answers are those given in Answer T12.

T14 (a) $x = \frac{1}{2}\left(6 \pm \sqrt{36 - 4}\right) = 3 \pm \sqrt{8}$.

Sum of roots $= (3 + \sqrt{8}) + (3 - \sqrt{8}) = 6 = -b/a$; product of roots $= (3 - \sqrt{8})(3 + \sqrt{8}) = 9 - 8 = 1 = c/a$.

(b) $x = \frac{1}{6}(9 \pm \sqrt{81 - 4 \times 3 \times 2}) = \frac{1}{6}(9 \pm \sqrt{57})$.

Sum of roots $= 3 = -b/a$; product of roots $= (1/36)(81 - 57) = 2/3 = c/a$.

In both of these cases we use $a^2 - b^2 = (a-b)(a+b)$ to find the product.

T15 (a) The discriminant $= (-20)^2 - 4 \times 4 \times 25 = 0$; there are two equal roots.

(b) The discriminant $= 1 - 4 \times 6 = -23$; there are no real roots.

(c) The discriminant $= 1 - 4 \times (-6) = 25$; there are two real roots.

T16 The proof is exactly the same as for $\frac{1}{2}\left(1 - \sqrt{-3}\right)$, except that the sign in front of the square root is changed wherever the root appears.

T17 (a) $x^2 = -25$; $x = \sqrt{-25} = \sqrt{25 \times (-1)} = \sqrt{25} \times \sqrt{-1}$
$= \pm 5i$

(b) From Equation 23,

$x = \frac{1}{2}\left(-1 \pm \sqrt{1 - 4 \times 6}\right) = \frac{1}{2}\left(-1 \pm i\sqrt{23}\right)$.

T18 One of your first trials should show that $x = -1$ is a solution. Then taking out the factor $(x+1)$ gives

$$x^3 + x^2 - 4x - 4 = (x+1)(x^2 - 4)$$
$$= (x+1)(x-2)(x+2)$$

and the roots are therefore -1, 2 and -2.

T19 The root is $x = 1.860\,81$.

T20 $x_1 = 0.450\,00$, $x_2 = 0.454\,44$, $x_3 = 0.453\,08$,
$x_4 = 0.453\,50$, $x_5 = 0.453\,37$, $x_6 = 0.453\,41$,
$x_7 = 0.453\,39$, $x_8 = 0.453\,40$, $x_9 = 0.453\,40$.

T21 The sequence is 1, 1.2, 1.248, 1.249 9968, 1.250 000 00. It is characteristic of this method that the number of correct figures in the answer doubles at each stage.

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Exit test answers

E1 $x = 4$. (Reread Subsection 2.1 if you had difficulty with this question.)

E2 If $y = 1/(x - 2)$, then $3y = 12$, $y = 4$. Then $x - 2 = 1/4$, and $x = 9/4$. (Reread Subsection 2.2 if you had difficulty with this question.)

E3 $x = -3$, $y = 2$. (Reread Subsection 2.3 if you had difficulty with this question.)

E4 Your graph(s) should show the straight line $y = 2x - 1$ and three lines $y = k - x$ corresponding to values of k of 2, 5 and 8. The required solutions are the coordinates of the points of intersection. (Of course, it is possible to work out the answers algebraically. To do so, it is sensible to first find the general solution in terms of k, namely $x = (1 + k)/3$ and $y = (2k - 1)/3$.) For the specified values of k: if $k = 2$, then $x = 1$, $y = 1$; if $k = 5$, then $x = 2$, $y = 3$; if $k = 8$, then $x = 3$, $y = 5$. (Reread Subsection 2.4 if you had difficulty with this question.)

E5 These graphs are shown in Figure 2. (Reread Subsection 3.1 if you had difficulty with this question.)

E6 (a) $(x - 2)(x + 7) = 0$; so $x = 2$ or -7; sum of roots is -5, product of roots is -14.

(b) $(2x - 3)(x - 9) = 0$; so $x = 3/2$ or 9; sum of roots is 21/2, product of roots is 27/2.

Note that the sum and product of the roots are consistent with the coefficients of the equation in each case. (Reread Subsection 3.2 if you had difficulty with this question.)

E7 (a) $x = 1 \pm \sqrt{2}$; (b) $x = -1$ or $2/3$. (Reread Subsections 3.2 and 3.3 if you had difficulty with this question.)

E8 The discriminant $= 4(r^2 - s)$. If the discriminant is *greater than* zero, there are two real roots; if the discriminant is *equal to* zero there is a repeated root. (Reread Subsection 3.3 if you had difficulty with this question.)

E9 In this case the discriminant is negative, and the roots may be written as $x = r \pm \sqrt{r^2 - s}$ or $x = r \pm i\sqrt{s - r^2}$. (Reread Subsection 3.4 if you had difficulty with this question.)

E10 $x = -1$, $+2$, or $+3$. (Reread Subsection 4.1 if you had difficulty with this question.)

E11 As Figure 8 indicates, the roots are approximately -0.6, 1.6, and 2.7. (Reread Subsection 4.2 if you had difficulty with this question.)

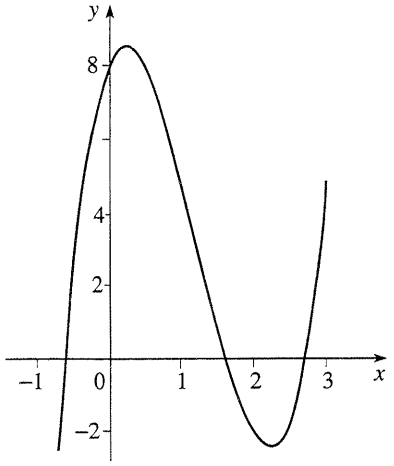


Figure 8 See Answer E11.

E12 Using the method of bisection with starting values 2.65 and 2.75 quickly leads to a value of 2.67 to two decimal places. In fact the root is at $x = 8/3$. (Reread Subsection 4.4 if you had difficulty with this question.)

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