Maths handbook

Introduction

The *Maths handbook* is intended as a quick reference to the mathematics introduced in the *FLAP* modules. It contains brief notes on the most commonly used symbols, definitions and results — you should consult the *Glossary* in order to find more details on any particular item.

This handbook is structured for easy reference; Section 0 *Units, symbols and statistics* refers to items that are introduced in the physics strand of *FLAP*, while the remaining sections are (loosely) associated with the correspondingly numbered blocks of modules that appear in the maths strand. Thus, Section 1 corresponds to the Block 1 mathematics modules, Section 2 to the Block 2 modules and so on. Within each section the material is arranged in a mathematically logical sequence with references to relevant modules provided in square brackets where appropriate.

In some instances you may find the same note arising in two or more sections reflecting the structure of the modules. Some material is split between sections, for example different aspects of the equations of lines are discussed in Sections 1 and 2 (and cross-referenced).

0 Units, symbols and statistics

0.1 SI base units

FLAP uses SI units. SI is an abbreviation for *Système International d'Unités*. The seven basic units are given in Table 1.

Table 1

Physical quantity	Name of unit	Symbol for SI unit	Physical quantity	Name of unit	Symbol for SI unit
length	metre	m	temperature	kelvin	K
mass	kilogram	kg	luminous intensity	candela	cd
time	second	s	amount of substance	mole	mol
electric current	ampere	A			

0.2 SI derived units

A number of derived units are given special names. The more common examples of these are given in Table 2.

Table 2

Physical quantity	Unit	Symbol	Definition
energy	joule	J	$\mathrm{kg\ m^2\ s^{-2}}$
force	newton	N	$\mathrm{kg\ m\,s^{-2} = J\ m^{-1}}$
power	watt	W	$\mathrm{kg\ m^2\ s^{-3} = J\ s^{-1}}$
electric charge	coulomb	C	A s
electric potential difference	volt	V	$\mathrm{kg\ m^2\ s^{-3}\ A^{-1} = J\ A^{-1}\ s^{-1}}$
electric resistance	ohm	Ω	$\mathrm{kg\ m^2\ s^{-3}\ A^{-2} = V\ A^{-1}}$
electric capacitance	farad	F	$\mathrm{A^2\ s^4\ kg^{-1}\ m^{-2} = A\ s\ V^{-1}}$
magnetic flux	weber	Wb	$\mathrm{kg\ m^2\ s^{-2}\ A^{-1} = V\ s}$
inductance	henry	H	$\mathrm{kg\ m^2\ s^{-2}\ A^{-2} = V\ s\ A^{-1}}$
magnetic field	tesla	T	$\mathrm{kg\ s^{-2}\ A^{-1} = Wb\ m^{-2}}$
frequency	hertz	Hz	$\mathrm{s^{-1}}$
pressure	pascal	Pa	$\mathrm{kg\ m^{-1}\ s^{-2} = N\ m^{-2} = J\ m^{-3}}$

Module M0.1

Introduction

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0.3 SI multiples

The standard SI multiples and submultiples are given in Table 3.

Table 3

Multiple	Prefix	Symbol for prefix	Multiple	Prefix	Symbol for prefix
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{0}			10^{-15}	femto	f

0.4 Non-SI units

A number of non-SI units are popular with scientists and continue to be prevalent in appropriate contexts. Some of these are listed in Table 4.

Table 4

Physical quantity	Unit	Symbol	Definition
mass	atomic mass unit	u	1.661×10^{-27} kg
length	light year	ly	9.462×10^{15} m
length	angstrom	Å	10^{-10} m
length	parsec	pc	3.086×10^{16} m
volume	litre	ℓ	10^{-3} m^3
time	year	yr	3.156×10^{7} s
temperature	degree Celsius	°C	1 K*
pressure	atmosphere	atm	1.013×10^{5} Pa
mass	tonne	t	10^{3} kg
energy	electronvolt	eV	1.602×10^{-19} J
energy	kilocalorie	kcal	4187 J
power	horsepower	hp	7.457×10^{2} W

* The zeros of the Kelvin and Celsius scales are different: 0 °C is equivalent to 273.15 K.

0.5 Angular units

The quantities associated with angular measurement have units which revert to being simple numbers when expressed in terms of the basic units. This follows from the definition of the angle θ (measured in radians) subtended at the centre of a circle of radius r, by an arc of length s, as $\theta/\text{rad} = s/r$. Angles may also be expressed in degrees of arc, and/or minutes and seconds of arc. To distinguish the scale being used, the appropriate unit must be appended to the simple number (Table 5).

Table 5

Physical quantity	Unit	Symbol	Definition
plane angle	radian	rad	$\theta/\text{rad} = s/r$
plane angle	degree of arc	°	$1° = (\pi/180)$ rad
plane angle	minute of arc	′	$1' = (1°/60)$
plane angle	second of arc	″	$1'' = (1°/3600)$

Angular values quoted without units should be assumed to be in rads.

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0.6 The Greek alphabet (and its pronunciation)

Table 6

Symbol		Name	Symbol		Name	Symbol		Name
lower case	upper case		lower case	upper case		lower case	upper case	
α*	A	alpha	ι	I	iota	ρ	P	rho (roe)
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau (taw)
δ	Δ	delta	μ	M	mu (mew)	υ	Y	upsilon
ε	E	epsilon	ν†	N	nu (new)	ϕ	Φ	phi (fie)
ζ	Z	zeta	ξ	Ξ	xi (csi)	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

* Be careful to distinguish α (Greek alpha) from $\propto$ ('proportional to') and a (italic a).

† Be careful also to distinguish ν (Greek nu) from υ (italic vee).

0.7 Mathematical symbols

$\propto$	proportional to		
$=$	equal to	$\equiv$	identical to
$\neq$	not equal to	$\approx$	approximately equal to
$>$	greater than	$<$	less than
$\geq$	greater than or equal to	$\leq$	less than or equal to
$>>$	much greater than	$<<$	much less than
$+ \; - \; \times \; \div$	basic operations; add, subtract, multiply, divide	$/$	alternative notation for division
$\pm$	plus or minus	...	and so on (ellipsis)
( )	parentheses	[ ]	square brackets
{ }	braces	$\in$	an element of (the set ...)
$\{x \in A: P\}$	the set of elements of A which have the property P	$\mathbb{N}$	the set of all natural numbers 1, 2, 3, ...
$\mathbb{Z}$	the set of all integers ..., −2, −1, 0, 1, 2, 3, ...	$\mathbb{Q}$	the set of all rational numbers
$\mathbb{R}$	the set of all real numbers	$\mathbb{C}$	the set of all complex numbers
e	the base of natural logarithms; e ≈ 2.718	π	pi, the ratio of a circle's circumference to its diameter; $\pi \approx 3.142$
∞	infinity	0!	zero factorial, defined to be 1
$r!$	r factorial, the number $r \times (r-1) \times (r-2) \ldots \times 3 \times 2 \times 1$	${}^nC_r$	the number of ways of choosing r items from n items irrespective of order $(= n!/(n-r)!\, r!)$
Σ	the sum of all terms of the immediately following kind	$\sum_{i=1}^{N}$	the sum of terms of the following kind for $i = 1$ to $i = N$
%	percent	(x, y, z)	Cartesian coordinates of a point
$\boldsymbol{v}$ or $\underset{\sim}{v}$	a vector (printed or handwritten)	(v_x, v_y, v_z)	ordered triple form of a vector

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$\mathbf{0}$	the zero vector		
$\lvert \boldsymbol{v} \rvert$	magnitude of the vector $\boldsymbol{v}$	$\overrightarrow{PQ}$	the displacement from P to Q, a directed line segment
$\hat{\boldsymbol{r}}$	a unit vector in the direction of $\boldsymbol{r}$	$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	Cartesian unit vectors
$\times$	vector (cross) product	$\cdot$	scalar (dot) product
i	square root of -1, $\sqrt{-1}$	$z = x + iy$	Cartesian form of a complex number
z^*	complex conjugate of z	$\lvert z \rvert$	modulus of z
$\arg(z)$	argument of z	$r\,e^{i\theta}$	exponential form of a complex number
$\rightarrow$	tends to (as in a limiting process)	$\lim_{h \to 0}$	the limit as h tends to zero
$f(x)$	f, a function of x	$f(a)$	the function f, evaluated at $x = a$
$f'(x)$	f prime, the derivative of $f(x)$	$f^{(n)}(x)$	the n^{th} derivative of $f(x)$
$x(t)$	x, a function of t	$\dot{x}(t)$	x dot, the first derivative of x with respect to t
$\ddot{x}(t)$	x double dot, the second derivative of x with respect to t	$\frac{df}{dx}(x)$	the first derivative of f with respect to x (same as $f'(x)$)
$\frac{df}{dx}(a)$	the first derivative of f with respect to x evaluated at $x = a$	$\frac{df(x)}{dx}$	alternative notation to $\frac{df}{dx}(x)$
$\frac{d^2 f(x)}{dx^2}$	the second derivative of f with respect to x	$\int f(x)\,dx$	the (indefinite) integral of $f(x)$ with respect to x
$\int_a^b f(x)\,dx$	the (definite) integral of $f(x)$ with respect to x, from a to b	$[F(x)]_a^b$	the difference $F(b) - F(a)$
$f(x, y)$	f, a function of x and y	$\lvert x \rvert$	the modulus (or absolute value) of x
$\frac{\partial f}{\partial x}$	the first partial derivative of $f(x, y)$ with respect to x	$\sqrt[n]{x}$	the n^{th} root of x
$\hat{\mathrm{p}}_x$	the operator representing p_x		
$\exp(x)$	the exponential function	e^x	alternative form for the exponential function
$\log_a(x)$	the logarithmic function to the base a		
$\log_{\mathrm{e}}(x)$	logarithmic function to the base e (sometimes written as $\ln x$, but not in *FLAP*)	$\log_{10}(x)$	logarithmic function to the base 10 (sometimes written as $\log x$, but not in *FLAP*)

parentheses are often omitted in the following expressions

$\sin(x)$	the sine function	$\cos(x)$	the cosine function
$\tan(x)$	the tangent function	$\operatorname{cosec}(x)$	the cosecant function
$\sec(x)$	the secant function	$\cot(x)$	the cotangent function
$\arcsin(x)$	the inverse sine function (sometimes written as $\sin^{-1} x$, but not in *FLAP*)	$\arccos(x)$	the inverse cosine function (sometimes written as $\cos^{-1} x$, but not in *FLAP*)

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arctan (x)	the inverse tangent function (sometimes written as $\tan^{-1}x$, but not in *FLAP*)	arccosec (x)	the inverse cosecant function (sometimes written as $\mathrm{cosec}^{-1}x$, but not in *FLAP*)
arcsec (x)	the inverse secant function (sometimes written as $\sec^{-1}x$, but not in *FLAP*)	arccot (x)	the inverse cotangent function (sometimes written as $\cot^{-1}x$, but not in *FLAP*)

0.8 Statistics

Arithmetic mean

The mean of n values, $x_1, x_2, x_3, x_4, \ldots, x_{n-2}, x_{n-1}, x_n$, of a quantity x

$$\langle x \rangle = \frac{x_1 + x_2 + x_3 + x_4 + \ldots x_{n-2} + x_{n-1} + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Standard deviation

Given n values, $x_1, x_2, x_3, x_4, x_5, \ldots x_n$, of a quantity x, with mean $\langle x \rangle$ (this is sometimes written as $\bar{x}$), the deviation of the i^{th} value is $d_i = x_i - \langle x \rangle$, and the standard deviation σ_n is defined by

$$\sigma_n = \sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_n^2}{n}} = \left(\frac{1}{n}\sum_{i=1}^{n} d_i^2\right)^{1/2}$$

According to *Bessel's correction*, if the values $x_1, \ldots, x_n$ are regarded as a sample of an infinite population of possible measurements, the best estimate of the standard derivation of that population is

$$\sigma = \left(\frac{n}{n-1}\right)^{1/2} \sigma_n$$

The distinction between σ and σ_n is insignificant for large n.

The method of least squares

The equation of the least squares line through the data points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) is given by $y = ax + b$

where $$a = \frac{\left(\sum_{i=1}^{n} x_i y_i\right) - n\langle x \rangle \langle y \rangle}{\left(\sum_{i=1}^{n} x_i^2\right) - n\langle x \rangle^2} \quad \text{and} \quad b = \langle y \rangle - a\langle x \rangle$$

The point $(\langle x \rangle, \langle y \rangle)$ lies on the least squares line.

Normal (or Gaussian) distribution

A quantity is said to be normally distributed if the probability (i.e. relative likelihood) that a single determination of its value will yield a result in the narrow range Δx, centred on the value x is

$$y\Delta x = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x - \langle x \rangle)^2 / 2\sigma^2] \Delta x$$

where $\langle x \rangle$ is the mean of the distribution and σ its standard deviation.

Given n readings of a normally distributed quantity, with (sample) mean $\langle x \rangle$ and (sample) standard deviation σ_n, then about 68% of the readings will lie within $\pm\sigma_n$ of the mean value, 95% within $\pm 2\sigma_n$, and 99.7% within $\pm 3\sigma_n$, provided n is sufficiently large.

An estimate of the *standard error in the mean*, s_m, is given by $s_m \approx \dfrac{\sigma_n}{\sqrt{n-1}}$.

The standard error in a sample mean is

$$s_m \approx \frac{\sigma}{\sqrt{n}}$$

Errors

Independent errors $\pm e_1, \pm e_2, \pm e_3, \ldots \pm e_n$ in a measured quantity will give rise to an overall error $\pm E$ of:

$$E = \sqrt{e_1^2 + e_2^2 + e_3^2 + \ldots + e_n^2} = (e_1^2 + e_2^2 + e_3^2 + \ldots + e_n^2)^{1/2}$$

For quantities A and B with independent errors ΔA and ΔB:

If $X = A + B$, or $X = A - B$, the error in X is given by:

$$\Delta X = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

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If $X = AB$, or $X = A/B$, the error in X is given by:

$$\Delta X/X = \sqrt{(\Delta A/A)^2 + (\Delta B/B)^2}$$

If $X = A^n$, the error in X is given by:

$$\Delta X/X = n(\Delta A/A)$$

If $X = kA$, where k is a constant, the error in X is given by:

$$\Delta X = k\,\Delta A$$

Straight-line graphs from data

Logarithms and other functions may be used to obtain straight-line fits to data and hence provide insight into the functional relation between the measured quantities x and y.

(In Table 7, m, n, C and A denote constants and g^{-1} is the inverse function to g.)

Table 7

Points plotted		Straight line ($Y = mX + C$)	Relation between x and y
$X = x$,	$Y = \log_e y$	$\log_e y = mx + \log_e A$	$y = A\,e^{mx}$
$X = \log_e x$,	$Y = \log_e y$	$\log_e y = m \log_e x + \log_e A$	$y = Ax^m$
$X = f(x)$,	$Y = g(y)$	$g(y) = mf(x) + C$	$y = g^{-1}(mf(x) + C)$
$X = x^n$,	$Y = y$	$y = mX + C$	$y = mx^n + C$

1 Algebra and functions

1.1 Arithmetic and algebra [M1.1]

Order of priority for basic mathematical operations: anything in brackets, multiplication ($\times$) and division ($/$), then addition ($+$) and subtraction ($-$)

In *FLAP* we use a slash ($/$) rather than the division symbol ($\div$) as the pages are designed to be photocopied and there is a chance that the latter symbol might be confused with the + sign.

Powers: a^b denotes a to the power b

Roots: $a^{p/q}$ denotes the q^{th} root of a^p, i.e. $\sqrt[q]{a^p}$

Reciprocals: $a^{-b} = \dfrac{1}{a^b} \quad (a \neq 0)$

$a^0 = 1$

$a^1 = a$

$a^m \times a^n = a^{m+n}$

$a^m / a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(a+b)^2 = a^2 + b^2 + 2ab$

$(a+b)(c+d) = ac + ad + bc + bd$

$\dfrac{a/b}{c/d} = \dfrac{ad}{bc}$

$\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+cd}{bd} \quad \left(not\ \dfrac{a+c}{b+d}\right)$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)(a-b) = a^2 - b^2$

The modulus or absolute value of a number p is denoted by $|p|$, and if p is a positive number then $|p| = p$ and $|-p| = p$.

1.2 Equality, proportionality and inequality [M1.1]

Proportionality

If $y \propto x$ then $y = Kx$ where K is a constant of proportionality.

Inequality

If $x > y$ and a is any number, then $x + a > y + a$

If $x > y$ and k is a positive number, then $kx > ky$

If $x > y$ and k is a negative number, then $kx < ky$

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1.3 Functions and graphs [M1.3]

A function is any combination of two sets (called the domain and the codomain) and a rule that meets the following conditions:

- the rule may be applied to every element of the first set,
- the rule associates a *single* element of the second set with each element of the first set.

If the first set consists of values of an independent variable x, and the second set consists of values of y (called the dependent variable) then we write $y = f(x)$ and say that y is a function of x.

Inverse functions

The inverse function of $f(x)$ is a function $g(y)$ such that if $y = f(x)$, then $x = g(y)$ for every value of x in the domain of $f(x)$. If $y = f(x)$, its inverse function is often denoted by f^{-1}, so that $x = f^{-1}(y)$.

Straight lines

A linear function is any function that may be written in the form

See also Subsection 2.2 and Figures 1 to 3.

$$f(x) = mx + c$$

where m and c are constants, called the gradient and intercept, respectively. Its graph is a straight line.

The equation of a straight line can be represented in various ways:

- gradient–intercept form $y = mx + c$
- point–gradient form $y - y_0 = m(x - x_0)$
- two-point form $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$

gradient, $m = \dfrac{\Delta y}{\Delta x} = \dfrac{\text{rise}}{\text{run}}$

Quadratics

A quadratic function is any function that may be written in the form

See also Subsection 2.3 and Figures 4 to 8.

$$f(x) = ax^2 + bx + c$$

where a, b and c are constants, and $a \neq 0$. Its graph is a parabola.

A quadratic function may be written in various ways:

- completed square form $f(x) = a(x - p)^2 + q$

  where $p = \dfrac{-b}{2a}$ and $q = \dfrac{-b^2}{4a} + c$

- factorized form $f(x) = a(x - \alpha)(x - \beta)$

  where $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ $(a \neq 0)$

Polynomials

A polynomial function of degree n is any function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2} + a_{n-1} x^{n-1} + a_n x^n$$

where n is an integer, and $a_n \neq 0$. The $n + 1$ constants $a_0, a_1, a_2, \ldots a_{n-2}, a_{n-1}$ and a_n are called the coefficients of the polynomial.

1.4 Solving equations [M1.4]

The quadratic equation $ax^2 + bx + c = 0$ has roots (i.e. solutions)

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$$

These are often written as $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- if $b^2 > 4ac$ there are two real roots
- if $b^2 = 4ac$ there is one real root
- if $b^2 < 4ac$ there are no real roots

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The roots satisfy $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

See Figures 6 to 8.

The fundamental theorem of algebra states that *every* polynomial equation of degree n has precisely n roots provided complex roots and repeated roots are counted.

(For numerical solutions of equations see the Newton–Raphson formula in Subsection 4.5.)

1.5 Exponential and logarithmic functions [M1.5]

Exponential functions

For more on derivatives of exp and log see Section 4.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828182845905\ldots \quad \text{(to 14 decimal places)}$$

$$\exp(x) = e^x \qquad \frac{d}{dx}(e^{kx}) = k e^{kx} \qquad e^0 = 1 \qquad e^1 = e \qquad e^{-1} = 1/e$$

See Figure 16.

$$e^x \times e^y = e^{x+y} \qquad e^x/e^y = e^{x-y} \qquad (e^x)^y = e^{xy}$$

a^x can be written as e^{kx} where $e^k = a$

Logarithmic functions

See Figure 17.

If $x = a^y$ then $y = \log_a x$ $(x > 0$ and $a > 0)$

$$\log_a(1) = 0 \qquad \log_a(a) = 1 \qquad \log_a(x^y) = y\log_a(x)$$

$$\log_a(x) + \log_a(y) = \log_a(xy) \qquad \log_a(a^x) = x$$

$$\log_a(x) - \log_a(y) = \log_a(x/y) \qquad \log_a(x) = \log_b(x)/\log_b(a)$$

Inverse relation of exp and $\log_e$

exp and $\log_e$ are inverse functions.

If $y = \exp(x)$ then $x = \log_e(y)$

$$\exp(\log_e x) = x$$

$$\log_e(e^x) = x$$

1.6 Trigonometric functions [M1.6]

Radian measure

Remember to set your calculator to radian mode or degree mode as appropriate.

$$\phi = \frac{s}{r}\,\text{rad} \qquad 2\pi \text{ radians} = 360°$$

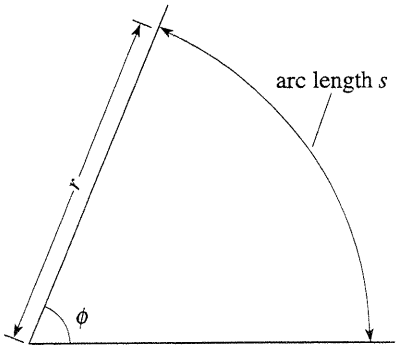


Figure A Radian measure.

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Pythagoras's theorem

For any right angled triangle $(\text{adjacent})^2 + (\text{opposite})^2 = (\text{hypotenuse})^2$

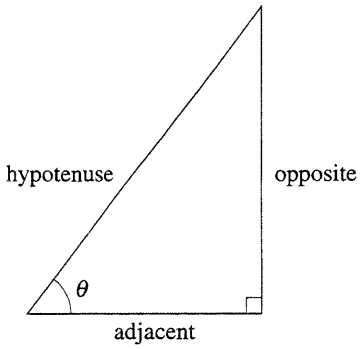


Figure B A right-angled triangle.

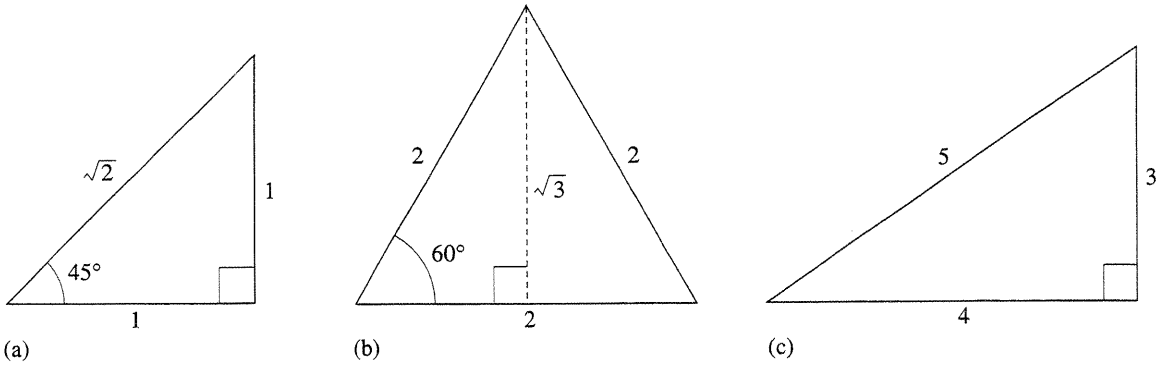


Figure C Useful triangles.

Trigonometric ratios

For a right-angled triangle, and for $0° \leq \theta \leq 90°$:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Table 8 Angles and their trigonometric ratios.

θ /degrees	θ /radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\pi/6$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
45	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90	$\pi/2$	1	0	undefined

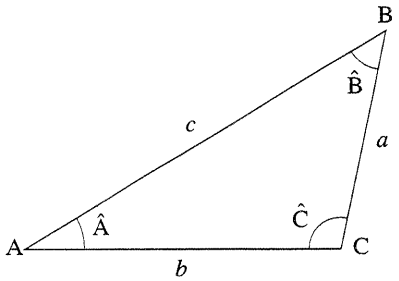


Figure D The sides and angles of an arbitrary triangle.

Sine rule

For any triangle with sides of length a, b, c:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}}$$

Beware of ambiguity when given two sides and an angle not between them, especially if the angle is opposite the shorter side.

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Cosine rule

For any triangle with sides of length a, b, c: $a^2 = b^2 + c^2 - 2bc\cos\hat{A}$ See Figure D.

Trigonometric functions and identities

The trigonometric functions, $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$, are periodic functions of period 2π (period π for $\tan(\theta)$) that are defined to agree with the corresponding trigonometric ratios in the range $0° \leq \theta \leq 90°$. They satisfy the following conditions. See Figures 18 to 20.

Table 9

Angular range	Positive trigonometric functions
$0° \leq \theta < 90°$	all
$90° \leq \theta < 180°$	$\sin\theta$
$180° \leq \theta < 270°$	$\tan\theta$
$270° \leq \theta < 360°$	$\cos\theta$

sin	all
positive	positive
tan	cos
positive	positive

Symmetry relations:

$\sin(-\alpha) = -\sin(\alpha)$

$\cos(-\alpha) = \cos(\alpha)$

$\tan(-\alpha) = -\tan(\alpha)$

Reciprocal trigonometric functions: See Figures 24 to 26.

$\sec(\alpha) = 1/\cos(\alpha)$

$\operatorname{cosec}(\alpha) = 1/\sin(\alpha)$

$\cot(\alpha) = 1/\tan(\alpha)$

Basic identities:

$\sin^2(\alpha) + \cos^2(\alpha) = 1$

$\sec^2(\alpha) = 1 + \tan^2(\alpha)$

$\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$

Addition formulae:

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \quad \text{if} \quad \alpha+\beta \neq \frac{(2n+1)\pi}{2}$$

$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} \quad \text{if} \quad \alpha-\beta \neq \frac{(2n+1)\pi}{2}$$

Double-angle formulae:

$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha) = 2\cos^2(\alpha) - 1$

Half-angle formulae:

If $t = \tan(\alpha/2)$ then

$$\sin(\alpha) = \frac{2t}{1+t^2} \quad \cos(\alpha) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \tan(\alpha) = \frac{2t}{1-t^2}$$

Sum formulae:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

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$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Product formulae:

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$-2\sin(\alpha)\sin(\beta) = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

Small angle approximations

For a small angle θ, measured in radians:

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \quad \text{and} \quad \sin\theta \approx \tan\theta \approx \theta$$

Table 10 Trigonometric functions between 0 and 2π.

θ/radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
$\pi/2$	1	0	undefined
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$3\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1
$5\pi/6$	1/2	$-\sqrt{3}/2$	$-1/\sqrt{3}$
π	0	-1	0
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3}$
$5\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
$3\pi/2$	-1	0	undefined
$5\pi/3$	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$
$7\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1
$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$
2π	0	1	0

Inverse trigonometric functions

$\arcsin(\sin\theta) = \theta$	$-\pi/2 \le \theta \le \pi/2$
$\arccos(\cos\theta) = \theta$	$0 \le \theta \le \pi$
$\arctan(\tan\theta) = \theta$	$-\pi/2 < \theta < \pi/2$
$\text{arccosec}(\text{cosec}\,\theta) = \theta$	$-\pi/2 \le \theta \le \pi/2,\ \theta \ne 0$
$\text{arcsec}(\sec\theta) = \theta$	$0 \le \theta \le \pi,\ \theta \ne \pi/2$
$\text{arccot}(\cot\theta) = \theta$	$0 < \theta < \pi$

The abbreviations asin (θ), acos (θ) and atan (θ) or alternatively $\sin^{-1}(\theta)$, $\cos^{-1}(\theta)$ and $\tan^{-1}(\theta)$, are sometimes used for the inverse trigonometric functions, but not in *FLAP*.

See Figures 21 to 23 and 27 to 29.

1.7 Summations and series [M1.7]

$$\sum_{i=1}^{N}(x_i + y_i) = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i$$

$$\sum_{i=1}^{N} a x_i = a\sum_{i=1}^{N} x_i$$

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$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1+4+9+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1+8+27+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$$

Arithmetic progression

$$\sum_{k=0}^{n-1}(a+kh) = \frac{n}{2}[2a+(n-1)h] = \frac{n}{2}(a+b) \quad \text{where } b = a+(n-1)h$$

Geometric progression

$$\sum_{k=0}^{n-1} ar^k = \frac{a(r^n-1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)} \qquad r \neq 1$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad |r| < 1$$

Binomial expansion

For any real numbers a and b and any positive integer n

This is sometimes called the binomial theorem.

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_{n-k}\, a^{n-k} b^k$$

$$= {}^nC_n a^n + {}^nC_{n-1} a^{n-1} b + {}^nC_{n-2} a^{n-2} b^2 + \ldots$$

$$\ldots + {}^nC_{n-r} a^{n-r} b^r + \ldots + {}^nC_1 a b^{n-1} + {}^nC_0 b^n$$

where $${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{r(r-1)(r-2)\ldots\times 2\times 1} = \frac{n!}{r!(n-r)!}$$

with $$r! = r\times(r-1)\times\ldots\times 3\times 2\times 1 \quad \text{and} \quad 0! = 1$$

$(n+1)! = (n+1)n!$

also $${}^nC_r = {}^nC_{n-r} = {}^{n-1}C_r + {}^{n-1}C_{r-1} \quad \text{with} \quad {}^nC_n = {}^nC_0 = 1$$

For any real r, not necessarily an integer:

See also Taylor series in Section 4

$$(1+x)^r = 1 + \frac{rx}{1!} + \frac{r(r-1)x^2}{2!} + \frac{r(r-1)(r-2)x^3}{3!} + \ldots \quad \text{for } -1 < x < 1$$

d'Alembert's ratio test

Given a series of positive terms $\sum_{n=1}^{\infty} a_n$ with $R = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$

- if $R < 1$ the series is convergent,
- if $R > 1$ the series is divergent,
- if $R = 1$ the test fails to identify whether the series is convergent, divergent or neither.

2 Geometry and vectors

2.1 Shapes, areas and volumes [M2.1]

The sum of the interior angles of an n sided polygon is $(2n-4)\times 90°$, i.e. $(2n-4)$ right angles.

Conditions for congruent triangles

Lengths of all three sides. (SSS)

Lengths of two sides and the angle between them. (SAS)

Length of one side and the angles at each end of it. (ASA)

A right angle, the hypotenuse and the length of one other side. (RHS)

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Areas and volumes

Table 11 Useful formulae for areas.

Figure	Surface area
rectangle	base × height
parallelogram	base × height
triangle	$\frac{1}{2}$ × base × height
	If $s = \frac{1}{2}(a+b+c)$ (see Figure D) the area can also be written in the form $\sqrt{s(s-a)(s-b)(s-c)}$
trapezium	$\frac{1}{2}\times$ (base$_1$ + base$_2$) × height
circle	πr^2
sector of a circle	$\frac{1}{2}\theta r^2$
ellipse	πab
cylinder (*not* including ends)	$2\pi r$ × height*
cone (*not* including base)	$\frac{1}{2}$ × base perimeter × slant height
sphere	$4\pi r^2$

*The circumference of a circle of radius r is $2\pi r$.

Table 12 Useful formulae for volumes.

Figure	Volume
rectangular prism	base area × height
cylinder	πr^2 × height
cone (or pyramid)	$\frac{1}{3}$ × base area × height
sphere	$\frac{4}{3}\pi r^3$

2.2 Basic coordinate geometry [M2.2]

Lines

General equation of a line $ax + by + c = 0$

The equation of a straight line can be represented in various ways:

- gradient–intercept form $y = mx + c$
- point–gradient form $y - y_0 = m(x - x_0)$
- two-point form $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$

gradient, $m = \dfrac{\Delta y}{\Delta x} = \dfrac{\text{rise}}{\text{run}}$

Two lines with gradients m_1 and m_2 are perpendicular if $m_1 m_2 = -1$.

The distance PQ between the points P(x_1, y_1) and Q(x_2, y_2) is given by

$$\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(See also Subsection 1.3 and Figures 1 to 3.)

Circles

Equation of a circle of radius R with centre at the origin

$$x^2 + y^2 = R^2$$

See Figure 13.

Equation of a circle of radius R with centre at the point (x_0, y_0)

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

See Figure 14.

Equation of the tangent to the circle $x^2 + y^2 = R^2$ at the point (x_1, y_1)

$$x_1 x + y_1 y = R^2$$

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Polar coordinates

Polar to Cartesian conversion:

See Figure G in Subsection 3.2.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Cartesian to polar conversion:

$$r^2 = x^2 + y^2, \ \sin\theta = y/r \ \text{and} \ \cos\theta = x/r \qquad r \neq 0$$

Coordinate geometry in three dimensions

The distance between two points $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

The equation of a plane $ax + by + cz = d$

The equations of a line in three dimensions $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$

2.3 Conic sections [M2.3]

Table 13 (See also Figures 13, 5, 15 and 9.)

Conic	Circle	Parabola	Ellipse	Hyperbola
eccentricity	$e = 0$	$e = 1$	$0 \le e < 1$	$e > 1$
focus directrix		$(a, 0)$ $x = -a$	$(\pm ae, 0)$ $x = \pm a/e$	$(\pm ae, 0)$ $x = \pm a/e$
standard equation	$x^2 + y^2 = a^2$ centre (0, 0) radius a	$y^2 = 4ax$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $b = a\sqrt{1 - e^2}$	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $b = a\sqrt{e^2 - 1}$
asymptotes	none	none	none	$y = \pm\dfrac{b}{a}x$
parametric form	$x = a\cos\theta$ $y = a\sin\theta$	$x = at^2$ $y = 2at$	$x = a\cos\theta$ $y = b\sin\theta$	$x = a\cosh\theta$ $y = b\sinh\theta$
tangent at (x_1, y_1)	$x_1x + y_1y = a^2$	$y_1y = 2a(x + x_1)$	$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$	$\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$
polar form	$r = a$, centre (0, 0), radius a $r = 2a\cos\theta$, centre $(a, 0)$, radius a	$\dfrac{L}{r} = 1 + \cos\theta$	$\dfrac{L}{r} = 1 + e\cos\theta$ $0 \le e < 1$	$\dfrac{L}{r} = 1 + e\cos\theta$ $e > 1$
some other forms and special cases	$(x-p)^2 + (x-q)^2 = a^2$ centre (p, q) radius a	$y = Ax^2 + Bx + C$		rectangular hyperbola $x^2 - y^2 = a^2$
	$x^2 + y^2 + 2Gx + 2Fy + C = 0$ centre $(-G, -F)$ radius $\sqrt{G^2 + F^2 - C}$			rectangular hyperbola $y = \dfrac{k}{x}$

2.4 Scalars and vectors [M2.4]

Representing scalars and vectors

A scalar quantity has magnitude alone. It can be specified by a single number (together with the appropriate units of measurement).

Vector quantities are physical quantities that have both magnitude and direction and which can be adequately represented by 'geometric' vectors (i.e. directed line segments).

In print vectors are usually denoted by bold type, e.g. $\boldsymbol{a}$. In handwritten work it is usual to use a wavy underline $\underset{\sim}{a}$.

Vectors may be represented algebraically in the following ways:

- ordered triple form $\boldsymbol{v} = (v_x, v_y, v_z)$
- Cartesian unit vector form $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$
- component vector form $\boldsymbol{v} = \boldsymbol{v}_x + \boldsymbol{v}_y + \boldsymbol{v}_z$

The Cartesian unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are defined below.

The magnitude of $\boldsymbol{v}$, is then the positive scalar quantity

$$|\boldsymbol{v}| = (|\boldsymbol{v}_x|^2 + |\boldsymbol{v}_y|^2 + |\boldsymbol{v}_z|^2)^{1/2} = (v_x^2 + v_y^2 + v_z^2)^{1/2}$$

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Given a vector $\boldsymbol{v}$, its orthogonal component vectors parallel and normal to a line inclined at an angle θ ($0° \leq \theta \leq 90°$) to $\boldsymbol{v}$ are of magnitude

$$|\boldsymbol{v}_p| = |\boldsymbol{v}| \cos\theta$$

and $$|\boldsymbol{v}_n| = |\boldsymbol{v}| \sin\theta$$

2.5 Working with vectors [M2.5]

Scaling a vector

For any vector $\boldsymbol{v}$ and scalar α $\quad |\alpha\boldsymbol{v}| = |\alpha||\boldsymbol{v}|$

- ordered triple form $\quad \alpha\boldsymbol{v} = (\alpha v_x, \alpha v_y, \alpha v_z)$
- Cartesian unit vector form $\quad \alpha\boldsymbol{v} = \alpha v_x\boldsymbol{i} + \alpha v_y\boldsymbol{j} + \alpha v_z\boldsymbol{k}$
- component vector form $\quad \alpha\boldsymbol{v} = \alpha\boldsymbol{v}_x + \alpha\boldsymbol{v}_y + \alpha\boldsymbol{v}_z$

Adding vectors

The triangle rule: Let vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ be represented by appropriate arrows (or directed line segments). If the arrow representing $\boldsymbol{b}$ is drawn from the head of the arrow representing $\boldsymbol{a}$, then an arrow from the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$ represents the vector sum $\boldsymbol{a} + \boldsymbol{b}$.

- ordered triple form $\quad \boldsymbol{a} + \boldsymbol{b} = ((a_x + b_x), (a_y + b_y), (a_z + b_z))$
- Cartesian unit vector form $\quad \boldsymbol{a} + \boldsymbol{b} = (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k}$
- component vector form $\quad \boldsymbol{a} + \boldsymbol{b} = (\boldsymbol{a}_x + \boldsymbol{b}_x) + (\boldsymbol{a}_y + \boldsymbol{b}_y) + (\boldsymbol{a}_z + \boldsymbol{b}_z)$

Properties of vector addition and scaling

The zero vector, $\boldsymbol{0}$, has the property that $\quad \boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$$

$$\boldsymbol{a} + (-1)\boldsymbol{a} = \boldsymbol{0}$$

$$\alpha(\boldsymbol{a} + \boldsymbol{b}) = \alpha\boldsymbol{a} + \alpha\boldsymbol{b} \qquad \text{for any scalar } \alpha$$

Unit vectors

A unit vector in the direction of the vector $\boldsymbol{a}$ is $\quad \hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$

Note that $|\hat{\boldsymbol{a}}| = 1$ $\quad$ (*Not* 1 unit, just 1)

The Cartesian unit vectors in the positive x, y and z directions are denoted $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

2.6 The scalar product [M2.6]

Definition and evaluation

The scalar (or dot) product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where θ (with $0 \leq \theta \leq \pi$) is the angle between the two vectors.

In terms of scalar components (see later)

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

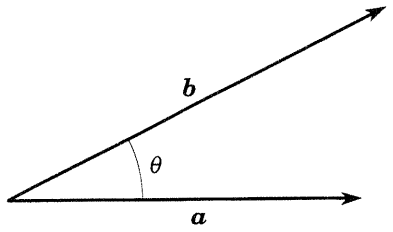


Figure E The angle between two vectors.

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Properties of the scalar product

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$$

$$\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos(0°) = |\boldsymbol{a}|^2$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$$

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{c}\cdot(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{c}\cdot\boldsymbol{a}+\boldsymbol{c}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c}$$

$(\alpha\boldsymbol{a})\cdot(\beta\boldsymbol{b}) = \alpha\beta(\boldsymbol{a}\cdot\boldsymbol{b})$ for any scalars α and β

If $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ and neither $\boldsymbol{a}$ are $\boldsymbol{b}$ are zero vectors, then $\boldsymbol{a}$ are $\boldsymbol{b}$ must be orthogonal (i.e. at right angles).

Scalar products of unit vectors

$$\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1 \quad \text{and} \quad \boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0$$

so $\boldsymbol{a}\cdot\boldsymbol{b} = (a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k})\cdot(b_x\boldsymbol{i}+b_y\boldsymbol{j}+b_z\boldsymbol{k}) = (a_xb_x+a_yb_y+a_zb_z)$

Angles, projections and scalar components

Angle between two vectors: $\cos\theta = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$

If $\boldsymbol{a} = (a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k})$ then $a_x = \boldsymbol{a}\cdot\boldsymbol{i}$, $a_y = \boldsymbol{a}\cdot\boldsymbol{j}$ and $a_z = \boldsymbol{a}\cdot\boldsymbol{k}$

2.7 The vector product [M2.7]

Definition and evaluation

The vector (or cross) product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by:

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\,\hat{\boldsymbol{n}}$$

where θ (with $0 \le \theta \le \pi$) is the angle between the two vectors and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

The unit vector $\hat{\boldsymbol{n}}$, and hence the vector product $\boldsymbol{a}\times\boldsymbol{b}$ points in the direction given by the right hand rule.

The right-hand rule: To determine the direction of $\boldsymbol{a}\times\boldsymbol{b}$, align the straightened fingers of your right hand with the vector $\boldsymbol{a}$ and turn your wrist until, by closing your palm, you can align your fingers with the vector $\boldsymbol{b}$. The direction of your extended thumb then shows the sense in which $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.

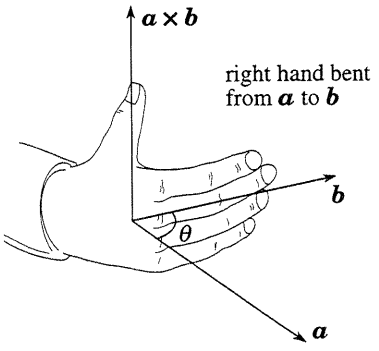


Figure F The right-hand rule.

In terms of scalar components (see later)

$$\boldsymbol{a}\times\boldsymbol{b} = (a_yb_z - b_ya_z,\ a_zb_x - b_za_x,\ a_xb_y - b_xa_y)$$

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Properties of the vector product

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$

$$\boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}$$

$$(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{b}\times\boldsymbol{c}$$

$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{c}$$

$$(\alpha\boldsymbol{a})\times\boldsymbol{b} = \alpha(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times(\alpha\boldsymbol{b}) \qquad \text{for any scalar } \alpha$$

Vector products, unit vectors and scalar components

$$\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0}$$

$$\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k} \quad \boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k}$$

$$\boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i} \quad \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i}$$

$$\boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j} \quad \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}$$

$$\boldsymbol{a}\times\boldsymbol{b} = (a_yb_z - b_ya_z)\boldsymbol{i} + (a_zb_x - b_za_x)\boldsymbol{j} + (a_xb_y - b_xa_y)\boldsymbol{k} \quad (\text{☞})$$

i.e. $$\boldsymbol{a}\times\boldsymbol{b} = (a_yb_z - b_ya_z,\ a_zb_x - b_za_x,\ a_xb_y - b_xa_y)$$

This is often remembered in terms of the *determinant*

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Cyclical property of the scalar triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

Vector triple product

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}) \quad \text{(the bac cab rule)}$$

and $$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$$

3 Complex numbers

3.1 Introducing complex numbers [M1.3]

Any complex number z may be written in the Cartesian form: $z = x + iy$

where: $i = \sqrt{-1}$

x is a real number, called the real part of z and written as $\mathrm{Re}(z)$

y is a real number, called the imaginary part of z and written as $\mathrm{Im}(z)$

Scaling a complex number

For any complex number z and any number λ

$$\lambda z = \lambda x + i\lambda y$$

Addition and multiplication of complex numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

$$z_1 + z_2 = (a + c) + i(b + d)$$

and $$z_1 \times z_2 = (ac - bd) + i(ad + bc)$$

If $z = x + iy$, then $$-z = -x - iy$$

and (if $z \neq 0$) $$z^{-1} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

The modulus and complex conjugate of a complex number

The modulus of $z = x + iy$ (written as $|z|$) is defined by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$$

The complex conjugate of z (written z^*) is defined by

$$z^* = x - iy = \mathrm{Re}(z) - i\mathrm{Im}(z)$$

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For arbitrary complex numbers z and w

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - z^*)$$

$$(zw)^* = z^*w^* \qquad (z^*)^* = z \quad \text{and} \quad |z|^2 = zz^*$$

$$zz^* = z^*z = x^2 + y^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = |z|^2$$

$$\frac{1}{z} = \frac{z^*}{|z|^2}$$

Expressing a quotient in Cartesian form (rationalizing)

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{x+iy}{a+ib} = \frac{x+iy}{a+ib} \times \frac{a-ib}{a-ib} = \frac{(x+iy)(a-ib)}{a^2+b^2} = \frac{(xa+yb)}{a^2+b^2} + \frac{i(ya-xb)}{a^2+b^2}$$

3.2 Polar representations [M3.2]

Polar form

Any complex number may be written in the polar form

$$z = r(\cos\theta + i\sin\theta)$$

where r and θ are real numbers (see Figure G). When written in polar form, the modulus of z is given by $|z| = r$, and θ is referred to as the argument of z (written as arg(z)). Adding 2π to the argument of a complex number does not change that complex number. The principal value of the argument of a complex number is the value θ which lies in the range $-\pi < \theta \leq \pi$.

To convert from polar form to Cartesian form:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

To convert from Cartesian form to polar form:

$$r = \sqrt{x^2 + y^2} \qquad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Just using $\tan\theta = y/x$ may lead to the wrong answer unless you also plot the polar form of the number on an Argand diagram (i.e. a complex plane).

If $z = x + iy$

then $$z = \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right)$$

i.e. $z = r(\cos\theta + i\sin\theta)$

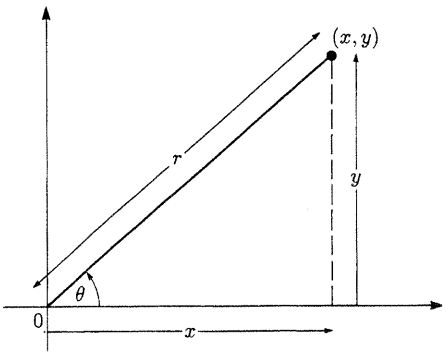


Figure G Polar representation of a complex number.

Exponential form

A complex number may also be written in exponential form

$$z = re^{i\theta}$$

thanks to Euler's formula

$$e^{i\theta} = r(\cos\theta + i\sin\theta)$$

If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, then

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$$z^* = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

3.3 Demoivre's theorem and complex algebra [M3.2]

The complex exponential function

If $z = x + iy$, then e^z is defined by

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

e^x and $\exp(x)$ are equivalent forms.

If z_1 and z_2 are complex numbers then, then

$$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2) \quad \text{and} \quad \exp(z_1)/\exp(z_2) = \exp(z_1 - z_2)$$

Demoivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

which may also be expressed as $(e^{i\theta})^n = e^{in\theta}$

Trigonometric identities

The following equations are often useful in proving trigonometric identities:

If $z = \cos\theta + i\sin\theta$ so that $\dfrac{1}{z} = \cos\theta - i\sin\theta$

then $z + \dfrac{1}{z} = 2\cos\theta$ $\qquad z - \dfrac{1}{z} = 2i\sin\theta$

$$z^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta) \qquad \frac{1}{z^n} = e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$$

The n^{th} roots of unity, $z_0, z_1, z_2, \ldots, z_{n-1}$ are roots of the equation $z^n - 1 = 0$ and are given by

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \quad k = 0, 1, 2, \ldots, (n-1)$$

Complex geometric series

The sum of a finite geometric series for complex numbers a and z is given by

$$a + az + az^2 + \ldots + az^n = a\frac{1 - z^{n+1}}{1 - z} \quad \text{if} \quad z \neq 1$$

and by $(n+1)a$ if $z = 1$.

Some results of complex algebra

If $z = re^{i\theta}$ $z_1 = r_1\exp(i\theta_1)$ and $z_2 = r_2\exp(i\theta_2)$ then

$\lambda z = \lambda re^{i\theta}$ for any real λ

$z_1z_2 = r_1r_2\exp[i(\theta_1 + \theta_2)]$

$|z_1z_2| = |z_1||z_2|$

$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$ (up to integer multiples of 2π)

$z_1/z_2 = (r_1/r_2)\exp[i(\theta_1 - \theta_2)]$ $\qquad z^n = r^ne^{in\theta}$

$z^{1/n} = r^{1/n}e^{i\theta/n}$ $\qquad z^{-1} = \dfrac{1}{z} = r^{-1}e^{-i\theta}$ $(r \neq 0)$

$z^* = re^{-i\theta}$ $\qquad zz^* = |z|^2$

$z^{-1} = \dfrac{1}{z} = \dfrac{z^*}{|z|^2}$ $(z \neq 0)$ $\qquad \left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$ $(z \neq 0)$

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4 Differentiation

4.1 Basic differentiation [M4.1 and M4.2]

Definition of the derivative

The derivative of the function $f(x)$ at $x = a$ is denoted by $\frac{df}{dx}(a)$ or $f'(a)$, and is defined by

$$f'(a) = \frac{df}{dx}(a) = \lim_{\Delta x \to 0}\left(\frac{f(a+\Delta x) - f(a)}{\Delta x}\right)$$

provided the limit exists and is unique.

If $y = f(x)$, the derivative $\frac{dy}{dx}$ is the *rate of change* of y with respect to x.

Standard derivatives

Table 14 Standard derivatives.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k (constant)	0	$\sec(kx)$	$k\sec(kx)\tan(kx)$
kx^n	nkx^{n-1}	$\cot(kx)$	$-k\,\mathrm{cosec}^2(kx)$
$\sin(kx)$	$k\cos(kx)$	$\exp(kx)$	$k\exp(kx)$
$\cos(kx)$	$-k\sin(kx)$	$\log_e(kx)$	$1/x$ (☞)
$\tan(kx)$	$k\sec^2(kx)$	a^x	$(\log_e a)\,a^x$
$\mathrm{cosec}(kx)$	$-k\,\mathrm{cosec}(kx)\cot(kx)$	$\log_a x$	$1/(x\log_e a)$

Note: k must be such that the function arguments are dimensionless.

$\frac{d}{dx}[\log_e(kx)]$ is independent of k.

Basic rules of differentiation

The sum rule

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

i.e. $(f(x)+g(x))' = f'(x)+g'(x)$

The constant multiple rule

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$$

i.e. $(kf)'(x) = kf'(x)$

The product rule

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$$

i.e. $(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$

The quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{(g(x))^2}$$

i.e.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

4.2 The chain rule and its uses [M4.3]

The chain rule: differentiating functions of functions

If y is a function of u, so $y = f(u)$, and if in turn u is a function of x, so $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = f'(u)\times g'(x)$$

Example

If $y = \sin(ax^2)$ we can write

$y = \sin(u)$ where $u = ax^2$

Then $\frac{dy}{dx} = \cos u \times 2ax$

i.e $\frac{dy}{dx} = 2ax\cos(ax^2)$

Implicit differentiation

An implicit function is a function that is defined by means of a relation such as $F(x, y) = 0$. If the equation $F(x, y) = 0$ can be written $g(x) = f(y)$ then, using the chain rule, we can say

$$\frac{dg}{dx} = \frac{df}{dy}\frac{dy}{dx}$$

If df/dy can be expressed in terms of x, this can be rearranged to give dy/dx in terms of x.

Example

If $y^3 = x^5$ then $3y^2\frac{dy}{dx} = 5x^4$

so $\frac{dy}{dx} = \frac{5x^4}{3y^2} = \frac{5}{3}x^{2/3}$

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The inversion rule

$$\frac{dx}{dy} = 1\Big/\left(\frac{dy}{dx}\right)$$

Parametric differentiation

If x and y are both defined in terms of a parameter θ, then the derivative $\frac{dy}{dx}$ may be determined by

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$$

4.3 Higher derivatives [M4.3]

If $y = f(x)$, then the n^{th} derivative of y is defined by

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \qquad n \text{ is an integer, } n > 1$$

The first four derivatives of $f(x)$ may also be represented by $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(\text{iv})}(x)$, and the n^{th} derivative by $f^{(n)}(x)$.

4.4 Stationary points and graph sketching [M4.4]

Derivatives and the shape of the graph

$f(x)$ is *increasing* throughout an interval if $\frac{df}{dx} > 0$ for all points in that interval.

$f(x)$ is *decreasing* throughout an interval if $\frac{df}{dx} < 0$ for all points in that interval.

$f(x)$ is *stationary* at any point where $\frac{df}{dx} = 0$, and such points are called *stationary points*.

If $f''(x) > 0$ throughout an interval the graph of $f(x)$ is concave upwards.

If $f''(x) < 0$ throughout an interval the graph of $f(x)$ is concave downwards.

Global maximum (or minimum) in an interval

In order to find the global maximum (or minimum) of a given function in an interval $a \le x \le b$:

(a) First ensure that the function is defined at every point of the interval. If this is not the case there may be no global maximum or minimum.

(b) Evaluate the function at every point where the derivative is undefined.

(c) Evaluate the function at the end points of the interval.

(d) Evaluate the function at the stationary points (i.e. points where the derivative of the function is zero).

The global maximum is the greatest, and the global minimum the least, of these values.

Local maxima and minima

To find the local maxima and minima of a given function, $f(x)$ say.

(a) If you can, find the stationary points (the points at which $f'(x) = 0$).

(b) If it is easy to find the second derivative, evaluate $f''(x)$ at the stationary points. Then:

(i) A positive second derivative indicates a local minimum.

(ii) A negative second derivative indicates a local maximum.

(iii) If the second derivative is zero at the stationary point one must investigate further:

If $f''(x)$ changes sign at the stationary point then the point is a point of inflection (with horizontal tangent).

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If $f''(x)$ does not change sign at the stationary point then either: $f''(x) > 0$ near the point, which means that the graph is *concave upwards*, and there is a local minimum; or, $f''(x) < 0$ near the point, which means that the graph is concave downwards, and there is a local maximum.

(c) If it is not easy to calculate $f''(x)$, examine the behaviour of $f'(x)$ near the stationary point.

(i) If $f'(x)$ changes sign at the stationary point then this point is a turning point, and must therefore be a local maximum or a local minimum. If $f'(x)$ changes from negative to positive there is a local minimum. If $f'(x)$ changes from positive to negative there is a local maximum.

(ii) If $f'(x)$ does not change sign at the stationary point then this point is a point of inflection.

(d) If for a given function the first and second derivative tests are too difficult to apply, useful information can often be obtained by finding the values of $f(x)$ at values of x that are close to the stationary point and by plotting the graph of $f(x)$ on a calculator.

Graph sketching

In order to sketch the graph of a function, the following points may be considered:

(a) The symmetry of the function.
Is it *even* ($f(-x) = f(x)$) or *odd* ($f(-x) = -f(x)$) ?

(b) Intervals or points where the function is undefined.

(c) Points at which the graph crosses the axes. Do the intersections on the x-axis correspond to multiple roots?

(d) Behaviour of the graph near points at which the function is undefined. Are there any asymptotes?

(e) Behaviour of the graph when $|x| >> 1$. Does the graph approach some simple curve?

(f) Behaviour of the graph where $|x| << 1$. What is the tangent at the origin?

(g) Points at which the function is stationary. Are these turning points (or points of inflection with a horizontal tangent)? Where is the function increasing and where is it decreasing?

(h) Where is the graph concave upwards and where is it *concave* downwards?

(i) Are there any *points of* inflection (i.e. points at which the second derivative of the function changes sign)?

4.5 Taylor series and Taylor polynomials [M4.5]

The Taylor polynomial of degree n for $f(x)$ near $x = 0$ is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

The Taylor series or expansion for $f(x)$ near $x = 0$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

The Taylor polynomial of degree n for $f(x)$ near $x = a$ is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylor series or expansion for $f(x)$ near $x = a$ is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

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Useful Taylor series

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \text{all real } x$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad \text{all real } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad \text{all real } x$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \qquad -\pi/2 < x < \pi/2$$

$$\log_e(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad -1 \le x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \qquad -1 < x < 1$$

For any real number r:

$$(1+x)^r = 1 + \frac{rx}{1!} + \frac{r(r-1)x^2}{2!} + \frac{r(r-1)(r-2)x^3}{3!} + \ldots \qquad -1 < x < 1$$

The Newton–Raphson formula

If x_n is the n^{th} approximation to the root of the equation $f(x) = 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{provided } f'(x_n) \neq 0$$

4.6 Hyperbolic functions and differentiation [M4.6]

Sinh, cosh and tanh

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

See Figures 30 to 32.

Relationship to the hyperbola

A hyperbola can be defined by means of the parametric equations

$$x = a\cosh(t) \qquad y = b\sinh(t)$$

Taylor expansions

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \qquad \text{all real } x$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad \text{all real } x$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots \qquad \text{all real } x$$

Derivatives

The derivatives of the two basic hyperbolic functions are

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

Note: $\frac{d}{dx}\cosh(x) \neq -\sinh(x)$

The trigonometric and hyperbolic functions

$$\sinh(ix) = i\sin(x) \qquad \cosh(ix) = \cos(x)$$

$$\sin(ix) = i\sinh(x) \qquad \cos(ix) = \cosh(x)$$

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Reciprocal hyperbolic functions

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \text{provided } x \neq 0$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)} \quad \text{provided } x \neq 0$$

Inverse hyperbolic functions

See Figures 33 to 35.

$$\operatorname{arcsinh}(\sinh(x)) = x$$

$$\operatorname{arccosh}(\cosh(x)) = x \quad \text{for } x \geq 0$$

$$\operatorname{arctanh}(\tanh(x)) = x$$

In terms of the logarithmic function

$$\operatorname{arcsinh}(x) = \log_e\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arccosh}(x) = \log_e\left(x + \sqrt{x^2 - 1}\right) \quad \text{for } x \geq 1$$

$$\operatorname{arctanh}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1$$

Identities

Symmetry relations:

$$\sinh(-x) = -\sinh(x), \quad \cosh(-x) = \cosh(x), \quad \tanh(-x) = -\tanh(x)$$

Basic identities:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \cosh^2(x) - \sinh^2(x) = 1$$

$$1 - \tanh^2(x) = \operatorname{sech}^2(x) \qquad \coth(x) = \frac{\operatorname{cosech}(x)}{\operatorname{sech}(x)} \quad \text{for } x \neq 0$$

Addition identities:

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

Double-argument identities:

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\cosh(2x) = 1 + 2\sinh^2(x)$$

$$\cosh(2x) = 2\cosh^2(x) - 1$$

$$\tanh(2x) = \frac{2\tanh(x)}{1 + \tanh^2(x)}$$

Half-argument identities:

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}[\cosh(x) + 1] \qquad \sinh^2\left(\frac{x}{2}\right) = \frac{1}{2}[\cosh(x) - 1]$$

and if $t = \tanh(x/2)$ then

$$\sinh(x) = \frac{2t}{1 - t^2}, \quad \cosh(x) = \frac{1 + t^2}{1 - t^2}, \quad \tanh(x) = \frac{2t}{1 + t^2}$$

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Sum identities:

$$\sinh(x)+\sinh(y)=2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\sinh(x)-\sinh(y)=2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\cosh(x)+\cosh(y)=2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\cosh(x)-\cosh(y)=2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

Product identities:

$$2\sinh(x)\cosh(y)=\sinh(x+y)+\sinh(x-y)$$

$$2\cosh(x)\cosh(y)=\cosh(x+y)+\cosh(x-y)$$

$$2\sinh(x)\sinh(y)=\cosh(x+y)-\cosh(x-y)$$

5 Integration

5.1 Introducing integration [M5.1]

Indefinite integrals

If $f(x)$ is a given function and $F(x)$ is a function such that its derivative $F'(x)=f(x)$ then $F(x)$ is known as an indefinite integral of $f(x)$, which we indicate by writing

$$\int f(x)\,dx=F(x)$$

A given function $f(x)$ may have many indefinite integrals, but any two of them may only differ by a constant, so in general:

$$\text{If}\quad \frac{dF}{dx}=f(x)\quad\text{then}\qquad \int f(x)\,dx=F(x)+C$$

where C is a constant of integration and $f(x)$ is called the integrand.

Definite integrals

The definite integral of $f(x)$ from $x=a$ to $x=b$ is defined as a limit of a sum

$$\int_a^b f(x)\,dx=\lim_{\Delta x\to 0}\left[\sum_{n=1}^{n} f(x_i)\,\Delta x_i\right]\quad\text{with}\quad \Delta x_i=x_{i+1}-x_i$$

where $x_1=a$, and $x_{n+1}=b$ and Δx is the largest of the intervals Δx_i.

Definite integrals may be evaluated using indefinite integrals thanks to the fundamental theorem of calculus which says that if $F(x)$ is an indefinite integral of $f(x)$ so that $F'(x)=f(x)$, then

$$\int_a^b f(x)\,dx=\left[F(x)\right]_a^b=F(b)-F(a)$$

5.2 Basic integration [M5.2]

Properties of definite integrals

1 $\int_a^a f(x)\,dx=0$

2 $\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$

3 $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx\qquad$ where $\quad a\le c\le b$

4 If k is any constant then $\int_a^b k\,f(x)\,dx=k\int_a^b f(x)\,dx$

5 If $f(x)\ge 0$ for all x in the interval $a\le x\le b$ then $\int_a^b f(x)\,dx\ge 0$

6 If $m\le f(x)\le M$ for all x in the interval $a\le x\le b$ then

$$m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$$

Technically, $f(x)$ must be a bounded function.

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7 $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

8 If $f(x) \geq g(x)$ for all values of x in the interval $a \leq x \leq b$ then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Standard integrals

Table 15 Some standard integrals (C is a constant)

Function $f(x)$	Indefinite integral $\int f(x)\,dx$	Function $f(x)$	Indefinite integral $\int f(x)\,dx$
k (a constant)	$kx + C$	$\sec(kx)$	$\frac{1}{k}\log_e\lvert\sec(kx)+\tan(kx)\rvert + C$
$kx^n, \quad n \neq -1$	$k\frac{x^{n+1}}{n+1} + C$	$\cot(kx)$	$\frac{1}{k}\log_e\lvert\sin(kx)\rvert + C$
$k\frac{1}{x}, \quad x \neq 0$	$k\log_e\lvert x\rvert + C$	$\log_e(kx)$	$x\log_e(kx) - x + C$
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + C$	e^{kx}	$\frac{1}{k}e^{kx} + C$
$\cos(kx)$	$\frac{1}{k}\sin(kx) + C$	$\frac{1}{a^2x^2+b^2}$	$\frac{1}{ab}\arctan\left(\frac{ax}{b}\right) + C$
$\tan(kx)$	$\frac{1}{k}\log_e\lvert\sec(kx)\rvert + C$	$\frac{1}{a^2x^2-b^2}$	$\frac{1}{2ab}\log_e\left\lvert\frac{ax-b}{ax+b}\right\rvert + C$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx) + C$	$\frac{1}{\sqrt{b^2+a^2x^2}}$	$\frac{1}{a}\operatorname{arcsinh}\left(\frac{ax}{b}\right) + C$
$\operatorname{cosec}(kx)$	$\frac{1}{k}\log_e\left\lvert\tan\left(\frac{kx}{2}\right)\right\rvert + C$	$\frac{1}{\sqrt{b^2-a^2x^2}}$	$\frac{1}{a}\arcsin\left(\frac{ax}{b}\right) + C$

Note: The constant k must be such that the arguments of the various functions are dimensionless. Also, $a > 0$.

5.3 Techniques of integration [M5.3]

Integration by parts

$$\int f(x)\,g(x)\,dx = F(x)g(x) - \int F(x)\frac{dg}{dx}\,dx, \text{ where } \frac{dF}{dx} = f(x)$$

Integration by substitution

An integral of the form $\int f(x)\,dx$ may often be transformed into a simpler integral by writing $x = g(y)$ and $dx = g'(y)\,dy$. Substituting these relations into the original integral gives a new integral in terms of the new variable y

$$\int f(x)\,dx = \int f(g(y))\,g'(y)\,dy$$

When applying this technique to a definite integral, transform the limits of integration along with the integration variable.

If an integral is recognized as being of the form $\int f(g(y))\,g'(y)\,dy$ it may be simpler to use the substitution $x = g(y)$.

Example: $\int \frac{dx}{\sqrt{1-x^2}}$

If $x = \sin y$ then $dx = \cos y\,dy$

and the integral may be written

$$\int \frac{1}{\sqrt{1-\sin^2 y}}\cos y\,dy$$

$$= \int dy = y + C = \arcsin x + C$$

Example: $\int \sin^3(y)\cos(y)\,dy$

If $x = g(y) = \sin y$ then $dx = \cos y\,dy$ and the integral may be written

$$\int x^3\,dx = \frac{x^4}{4} + C = \frac{1}{4}\sin^4(y) + C$$

5.4 Applications of integration [M5.4]

Area under a graph

The (signed) area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx$$

This definition implies that the area may be negative if $f(x) < 0$, or if $b < a$. Such an area is sometimes referred to as a signed area. Other texts may use an alternative definition in which the area is always positive.

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The area between two graphs

If the graphs of two functions $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$ (where $a < b$), the (positive) area between the graphs is given by

$$\int_a^b |f(x) - g(x)|\,dx$$

Length of an arc

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Solids of revolution

If a solid is obtained by rotating the area under the graph of $f(x)$ between $x = a$ and $x = b$ about the x-axis, then its volume V (known as a volume of revolution) is given by

$$V = \pi\int_a^b [f(x)]^2\,dx$$

and the area of its surface A (known as a surface of revolution) is given by

$$A = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

The total mass of a solid of varying density

Integration may be used to find the mass M of a solid whose density ρ varies with position. In the case of a rod of constant cross-sectional area A and length L, lying along the x-axis between 0 and L,

$$M = A\int_0^L \rho(x)\,dx$$

For a disc of height h and radius R, whose density $\rho(r)$ depends only on distance r from the axis,

$$M = 2\pi h\int_0^R r\rho(r)\,dr$$

For a sphere of radius R, whose density $\rho(r)$ depends only on distance r from the centre,

$$M = 4\pi\int_0^R r^2\rho(r)\,dr$$

Centre of mass

The position of the centre of mass of a solid object may be found by integration; for example, for a rod of mass M with constant cross-sectional area A and length L, lying along the x-axis between 0 and L, the x-coordinate of the centre of mass is

$$x_{cm} = \frac{A}{M}\int_0^L x\rho(x)\,dx$$

Average value of a function

The average value of a function $f(x)$ over the interval $a \leq x \leq b$ is defined as

$$f_{av} = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

Distributions and densities

A distribution (or density function) is a function specifying the fraction of a population for which a certain property has values lying in a small range between x and $x + \Delta x$; the distribution $f(x)$ is defined so that this fraction is equal to $f(x)\,\Delta x$.

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Such distributions satisfy a normalization condition $\int_a^b f(x)\,dx = 1$

and the mean of the distribution, $\langle x \rangle$, is given by $\langle x \rangle = \int_a^b x f(x)\,dx$

5.5 Further integration [M5.5]

This subsection only aims to introduce some of the more specialized techniques of integration. You should look elsewhere for more detailed treatments.

Partial fractions and completing the square

Integrals of the form $\int \frac{p(x)}{q(x)}dx$, where $p(x)$ and $q(x)$ are polynomials in x, may often be converted into sums of simpler integrals by writing the integrand in terms of its partial fractions. For example,

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

and

$$\frac{1}{(x+1)^2(x+2)} = \frac{-x}{(x+1)^2} + \frac{1}{(x+2)} = -\frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)}$$

If the method of partial fractions is to be useful, the denominator $q(x)$ must factorize, at least partially. If $q(x)$ is a quadratic function of x which does not factorize, the technique of writing $q(x)$ in completed square form and making a substitution of the form $x = \tan u$ allows us to find the integral. For example,

$$\int \frac{1}{x^2+4x+8}dx = \int \frac{1}{(x+2)^2+4}dx = \frac{1}{2}\arctan\left[\frac{x+2}{2}\right] + C$$

Completing the square in the denominator, and making a substitution of the form $x = \sin u$, allows us to find integrals of the form

$$\int \frac{1}{\sqrt{a+bx-cx^2}}dx \quad \text{where } c > 0$$

Splitting the numerator

Integrals of the form

$$\int \frac{px+q}{\sqrt{a+bx-cx^2}}dx$$

and (in the case where the denominator does not factorize)

$$\int \frac{px+q}{ax^2+bx+c}dx$$

can be simplified by writing the numerator in the form

$$\text{numerator} = \begin{pmatrix}\text{multiple of derivative of the quadratic} \\ \text{function in denominator}\end{pmatrix} + \text{constant}$$

the integral may then be written as the sum of two simpler integrals.

Hyperbolic functions

Two standard integrals, which can be found by substitution of the hyperbolic functions $x = \sinh u$ and $x = \cosh u$, respectively, are

$$\int \frac{1}{\sqrt{x^2+1}}dx = \operatorname{arcsinh} x + C \quad \text{and} \quad \int \frac{1}{\sqrt{x^2-1}}dx = \operatorname{arccosh} x + C$$

Integrals of the form

$$\int \frac{1}{\sqrt{A+Bx^2}}dx \quad \text{and} \quad \int \frac{1}{\sqrt{Bx^2-A}}dx$$

where A, B are positive constants, can be found by the substitution $x = a\sinh u$ or $x = a\cosh u$, where a is a suitably chosen constant.

Reduction formulae

Reduction formulae are equations that relate an integral involving a power of x (or some function of x) to an integral of the same form but involving a lower power of x

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(or related function of x). Two examples are

$$\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Improper integrals

Definite integrals in which one or both of the limits of integration is infinite are known as improper integrals. They are to be found by treating the infinite limit as a very large number and allowing it to tend to infinity

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \quad \text{and} \quad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

Gaussian integrals

Gaussian integrals are definite integrals of the form

$$\int_0^\infty x^{2n}\exp(-ax^2)\,dx \qquad (a>0,\ n \text{ an integer})$$

They may all be evaluated in terms of the basic Gaussian integral,

$$\int_0^\infty \exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

using the reduction formula

$$\int_0^\infty x^{2n}\exp(-ax^2)\,dx = \frac{2n-1}{2a}\int_0^\infty x^{2n-2}\exp(-ax^2)\,dx$$

Other useful definite integrals include:

$$\int_0^\infty x^n \exp(-ax)\,dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty x^{2n+1}\exp(-ax^2)\,dx = \frac{n!}{2a^{n+1}}$$

Line integrals

A line integral of the form $\int_{r_a}^{r_b} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$ is defined as a limit of a sum along a particular path. It may be evaluated by using the equation of the path to convert the line integral into an ordinary definite integral.

6 Differential equations

6.1 Introducing differential equations [M6.1]

Order and degree

The order of a differential equation is the order of the highest derivative in the equation. The degree of a differential equation is the highest power to which the highest order derivative in the equation is raised.

Linear and non-linear equations

A linear differential equation of order n is an equation of the form

$$a(x)\frac{d^n y}{dx^n} + b(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + g(x)\frac{dy}{dx} + h(x)y = f(x)$$

Non-linear differential equations may often be linearized by making appropriate approximations, but the resulting linear equation may have solutions that are qualitatively different from the solutions of the original non-linear equation.

General and particular solutions

The general solution of a linear differential equation of order k involves k arbitrary (essential) constants.

A particular solution is a solution that contains no arbitrary constants.

Initial and boundary conditions

Initial conditions (usually conditions that apply when the independent variable is zero), in which the dependent function and its derivatives are given at any *common*

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value of the independent variable, may be used to determine a particular solution, and then to predict the values of the dependent variable. Boundary conditions which are given at different values of the independent variable may be used for the same purpose. (Given conditions do not always lead to a unique solution.)

6.2 Solving first-order differential equations [M6.2]

First-order linear differential equations of the form $dy/dx = f(x)$ can be solved at once by direct integration, to give $y = \int f(x)\,dx$.

Provided the integral exists.

Equations of the form $dy/dx = h(y)$ can be solved by applying the inversion rule, $dy/dx = 1/(dx/dy)$, to obtain $dx/dy = 1/h(y)$, which may be directly integrated. This is a special case of the following, more general, technique.

Separation of variables

Equations of the form $dy/dx = f(x)h(y)$ may be rewritten as

$$\frac{1}{h(y)}\frac{dy}{dx} = f(x)$$

both sides can then be integrated with respect to x to give

$$\int \frac{1}{h(y)}\,dy = \int f(x)\,dx$$

it may then be possible to evaluate this and rearrange to find y.

Integrating factors

Some linear first-order differential equations with constant coefficients of the form $a(dy/dx) + by = f(x)$ can be solved by dividing both sides by a, multiplying both sides by an integrating factor $\exp(bx/a)$, and integrating with respect to x to obtain

$$ye^{bx/a} = \frac{1}{a}\int f(x)e^{bx/a}\,dx$$

Integrating factors may also be found found in more general cases, but this is beyond the scope of *FLAP*.

6.3 Solving second-order differential equations [M6.3]

Direct integration

Equations of the form $\dfrac{d^2x}{dt^2} = f(t)$ can be solved by direct integration.

Provided the integral exists.

More general method

The general linear second-order differential equations with constant coefficients is of the form

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = f(t)$$

Physical examples of this kind often involve t as the independent variable and x as the dependent variable. Our notation reflects this.

where a, b and c are real constants and $f(t)$ is independent of x.

If $f(t) = 0$ for all values of t the equation is said to be homogeneous, otherwise the equation is said to be inhomogeneous.

The general solution expresses x in terms of t, the given constants that appear in the equation and two additional arbitrary constants. The values of the arbitrary constants cannot be determined from the equation itself but must be found from boundary conditions or initial conditions as appropriate.

The general solution of the inhomogeneous equation is the sum of two parts, a particular solution $x_p(t)$, which may be any solution that does not contain arbitrary constants, and a complementary function $x_c(t)$ which does contain two arbitrary constants, and which satisfies the corresponding homogeneous equation

$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0$$

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Finding the complementary function

1 Given the values of a, b and c in the homogeneous equation, write down the auxiliary equation

$$ap^2 + bp + c = 0$$

2 Find the solutions of this quadratic equation, p_1 and p_2

3 (a) If $b^2 > 4ac$ the solutions will be two different real numbers and the complementary function will be

$$x_c(t) = B\exp(p_1 t) + C\exp(p_2 t)$$

The arbitrary constants B and C are determined using initial or boundary conditions.

(b) If $b^2 < 4ac$ the solutions will be two different complex numbers which may be written

$$p_1 = -2\gamma + i\omega \quad \text{and} \quad p_2 = -2\gamma - i\omega$$

where $\gamma = b/a$ and $\omega = \sqrt{\dfrac{c}{a} - \dfrac{\gamma^2}{4}}$

and the complementary function will be

$$x_c(t) = e^{-\gamma t/2}[B\cos(\omega t) + C\sin(\omega t)]$$

(c) If $b^2 = 4ac$ the solutions will be two equal real numbers $p_1 = p_2 = -b/2a = -\gamma/2$, and the complementary function will be

$$x_c(t) = (B + Ct)\,e^{-\gamma t/2}$$

Finding a particular solution

Determining a particular solution is generally much more difficult, and usually involves educated guesswork. For instance, if $f(t)$ has the general form

$$f(t) = f_0 \sin(\Omega t)$$

then the particular solution will be

$$x_p(t) = A\sin(\Omega t - \delta)$$

where $A = \dfrac{f_0}{\sqrt{(c/a - \Omega^2)^2 + (\gamma\Omega)^2}}$ and $\delta = \arctan\left(\dfrac{\gamma\Omega}{c/a - \Omega^2}\right)$

Note that the constants A and δ appearing in the particular solution are *not* arbitrary constants; their values are determined by the given values of a, b, c, f_0 and Ω and not by any initial or boundary conditions.

6.4 Partial derivatives and partial differential equations [M6.4]

Partial derivatives

If $f(x, t)$ is a function of the two independent variables x and t, the first partial derivatives of $f(x, t)$ is defined by

$$\left(\frac{\partial f}{\partial x}\right)_t = \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x, t) - f(x,t)}{\Delta x}\right]$$

and

$$\left(\frac{\partial f}{\partial t}\right)_x = \lim_{\Delta t \to 0}\left[\frac{f(x, t + \Delta t) - f(x,t)}{\Delta t}\right]$$

Partial derivatives may be evaluated by treating one variable as a constant, while differentiating in the ordinary way with respect to the other variable.

The second partial derivatives of $f(x, t)$ are defined by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial^2 f}{\partial t^2}, \quad \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial t \partial x}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial^2 f}{\partial x \partial t}$$

For well-behaved functions $\dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t}$

Partial differential equations

The one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

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where v is a constant has solutions of the general form $y = f(x - vt) + g(x + vt)$, including, as a special case $y(x,t) = A\sin[k(x - vt) + \phi]$, where A, $k\ (= 2\pi/\lambda)$ and ϕ are constants.

The one-dimensional time-dependent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

In the special case that $U(x, t)$ is independent of t, so that it may be written $U(x)$, this equation has solutions of the form

$$\Psi(x,t) = \psi(x)\exp(-iEt/\hbar)$$

where $\psi(x)$ satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Both the wave equation and the time-dependent Schrödinger equation are homogeneous linear equations and consequently have the property that if $f_1(x, t)$ and $f_2(x, t)$ are both solutions of one of the equations, then the linear combination $af_1(x, t) + bf_2(x, t)$ will also be a solution of that equation, where a and b are constants.

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Graphs, curves and functions

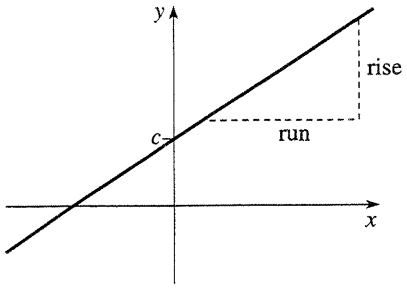


Figure 1 $y = mx + c$ where $m = \dfrac{\text{rise}}{\text{run}}$.

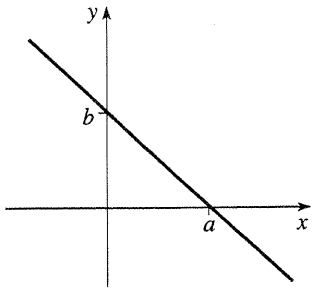


Figure 2 $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

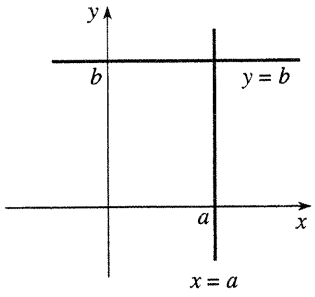


Figure 3 Horizontal line $y = b$ and vertical line $x = a$.

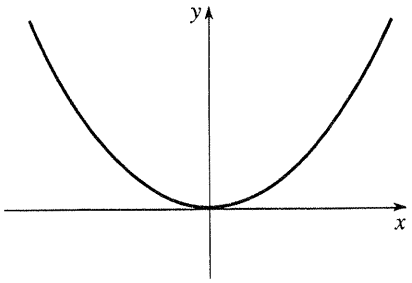


Figure 4 $y = x^2$.

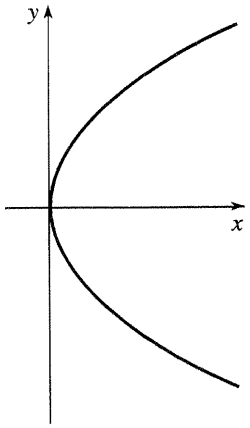


Figure 5 $y^2 = 4ax$.

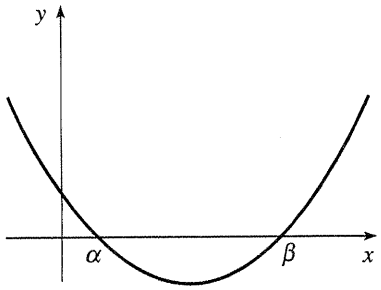


Figure 6 $y = (x - \alpha)(x - \beta)$.

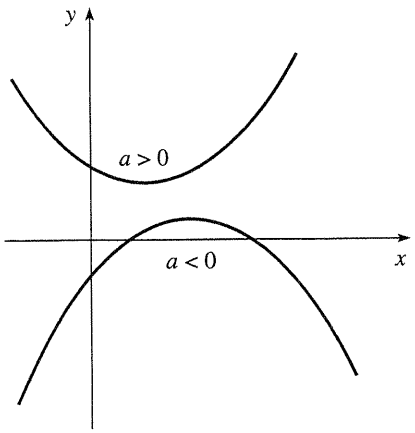


Figure 7 $y = ax^2 + bx + c$ with $a > 0$ (above) and $a < 0$ (below).

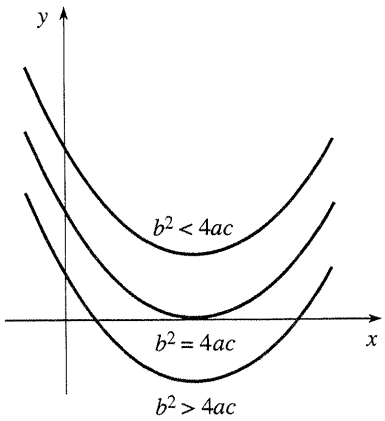


Figure 8 $y = ax^2 + bx + c$.

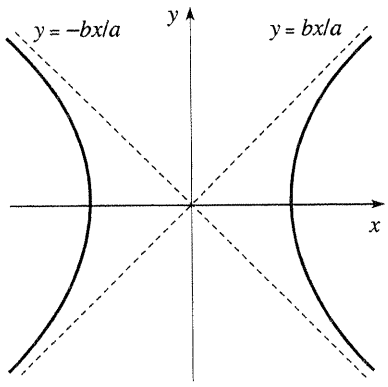


Figure 9 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

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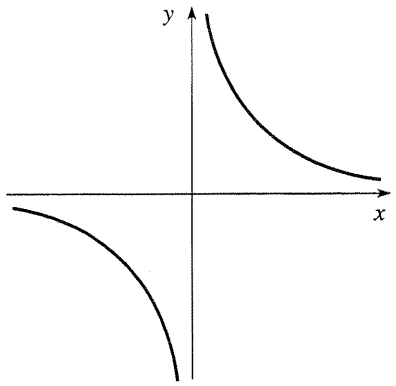


Figure 10 $y = 1/x$.

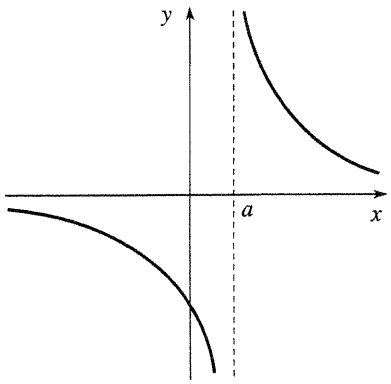


Figure 11 $y = \dfrac{1}{x-a}$.

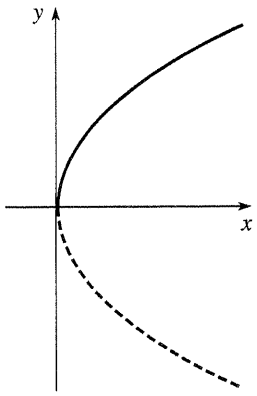


Figure 12 $y = \sqrt{x}$ and $y = -\sqrt{x}$ (dashed).

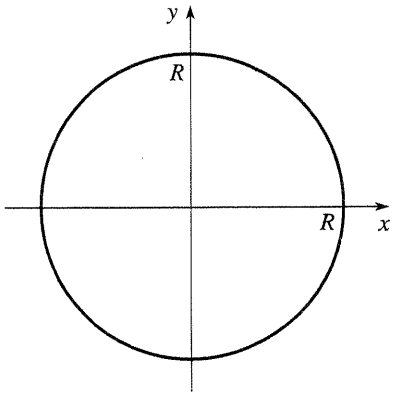


Figure 13 $x^2 + y^2 = R^2$.

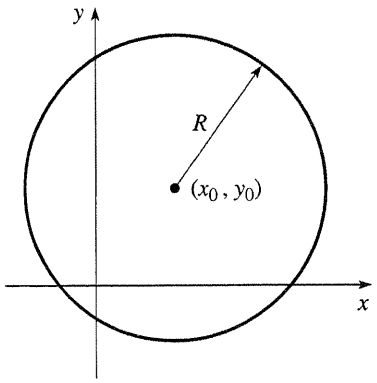


Figure 14 $(x - x_0)^2 + (y - y_0)^2 = R^2$.

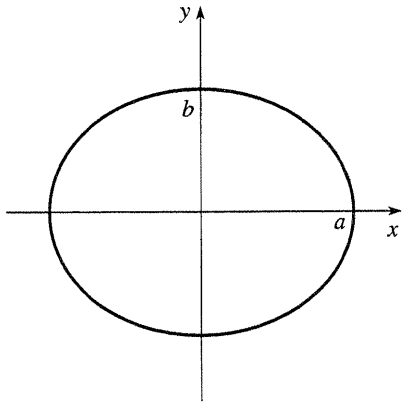


Figure 15 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

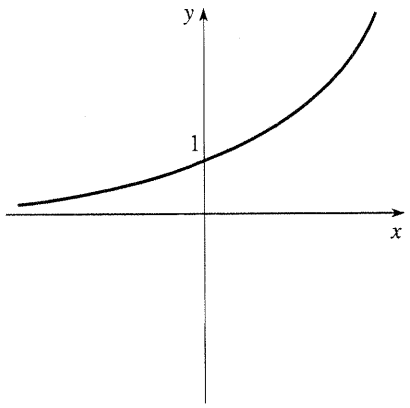


Figure 16 $y = a^x$.

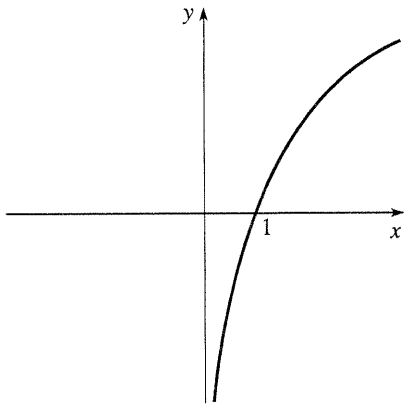


Figure 17 $y = \log_a x$.

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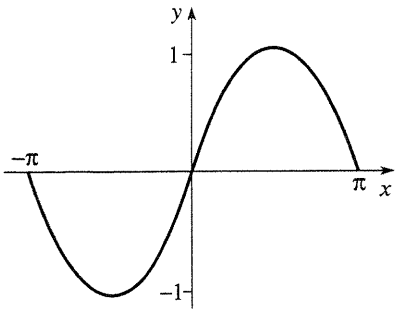


Figure 18 $y = \sin x$.

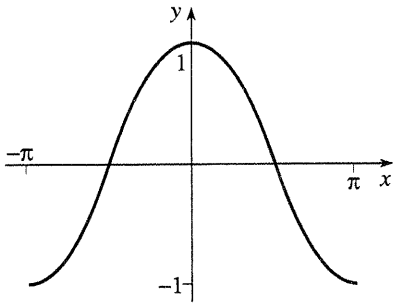


Figure 19 $y = \cos x$

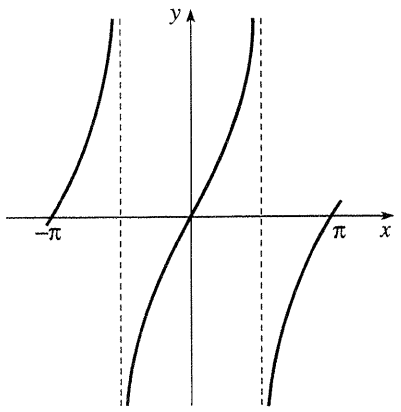


Figure 20 $y = \tan x$.

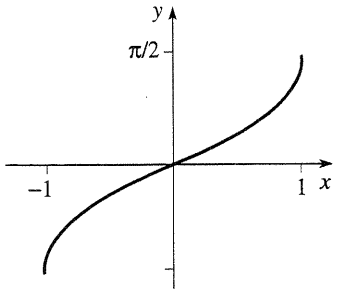


Figure 21 $y = \arcsin x$.

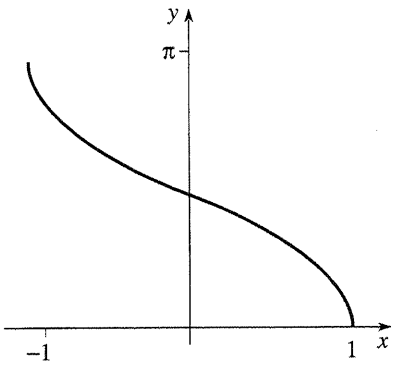


Figure 22 $y = \arccos x$.

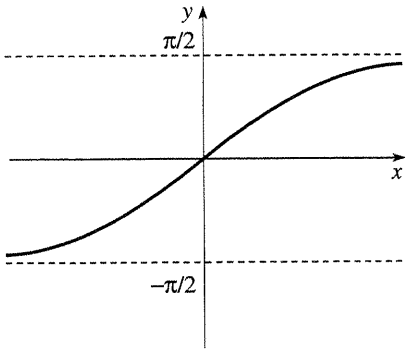


Figure 23 $y = \arctan x$.

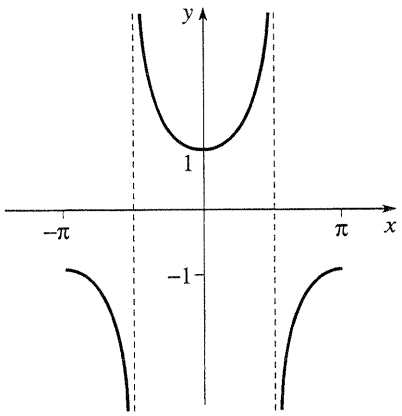


Figure 24 $y = \sec x$.

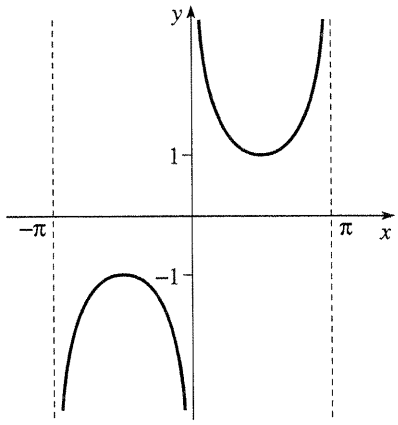


Figure 25 $y = \operatorname{cosec} x$.

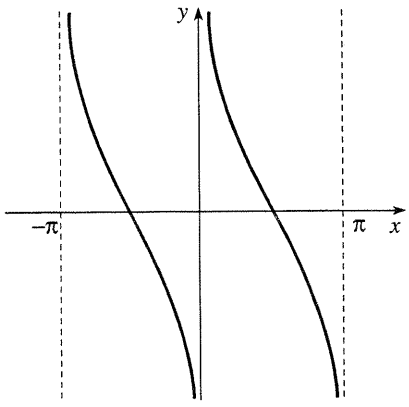


Figure 26 $y = \cot x$.

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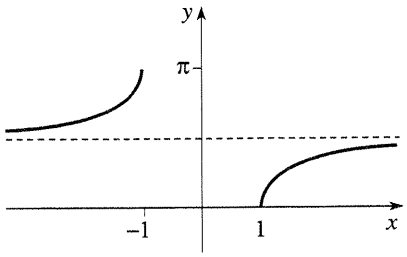


Figure 27 $y = \text{arcsec}\, x$.

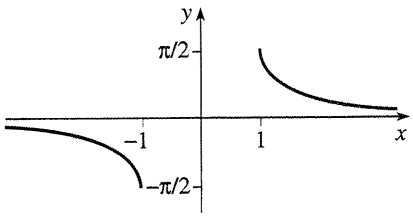


Figure 28 $y = \text{arccosec}\, x$.

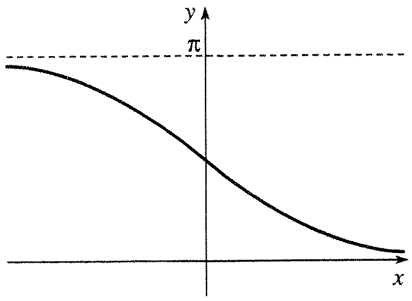


Figure 29 $y = \text{arccot}\, x$.

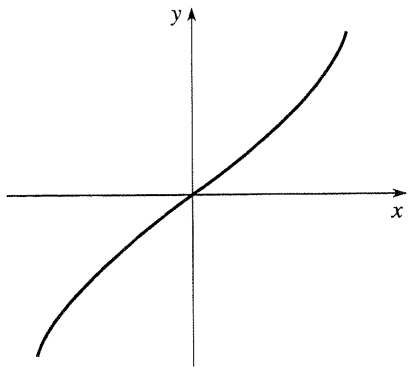


Figure 30 $y = \sinh x$.

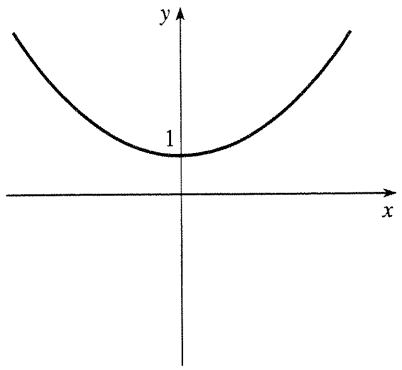


Figure 31 $y = \cosh x$.

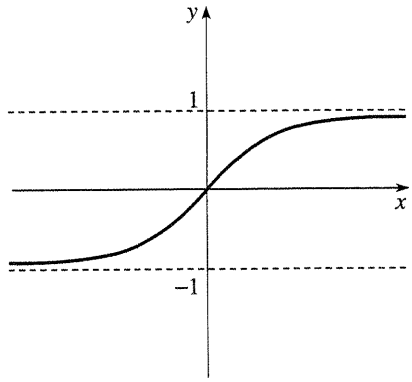


Figure 32 $y = \tanh x$.

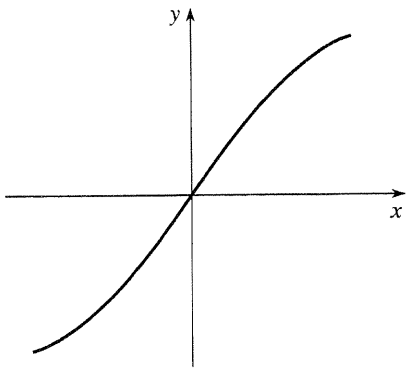


Figure 33 $y = \text{arcsinh}\, x$.

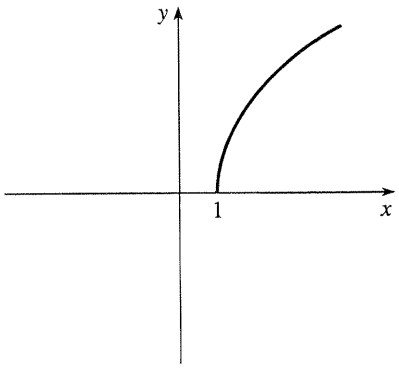


Figure 34 $y = \text{arccosh}\, x$.

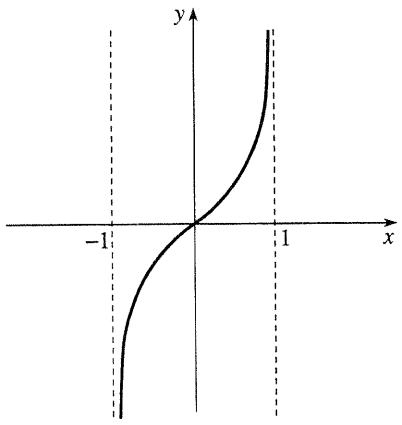


Figure 35 $y = \text{arctanh}\, x$.

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Table of constants

Useful constants

magnitude of the acceleration due to gravity	$g = 9.81\ \mathrm{m\ s^{-2}}$
gravitational constant	$G = 6.673 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$
Boltzmann's constant	$k = 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
Avogadro's constant	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
molar gas constant	$R = 8.314\ \mathrm{J\ K^{-1}\ mol^{-1}}$
Stefan's constant	$\sigma = 5.670 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
permittivity of free space	$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$ $1/4\pi\varepsilon_0 = 8.988 \times 10^{9}\ \mathrm{N\ m^2\ C^{-2}}$
permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\ m\ A^{-1}}$
speed of light in vacuum	$c = 2.998 \times 10^{8}\ \mathrm{m\ s^{-1}}$
Planck's constant	$h = 6.626 \times 10^{-34}\ \mathrm{J\ s}$
	$\hbar = h/2\pi = 1.055 \times 10^{-34}\ \mathrm{J\ s}$
Rydberg constant	$R = 1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Bohr radius	$a_0 = 5.292 \times 10^{-11}\ \mathrm{m}$
charge of proton	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
charge of electon	$-e = -1.602 \times 10^{-19}\ \mathrm{C}$
electron rest mass	$m_e = 9.109 \times 10^{-31}\ \mathrm{kg}$
electron charge to mass ratio	$-e/m_e = -1.759 \times 10^{11}\ \mathrm{C kg^{-1}}$
proton rest mass	$m_p = 1.673 \times 10^{-27}\ \mathrm{kg}$
neutron rest mass	$m_n = 1.675 \times 10^{-27}\ \mathrm{kg}$
mass of the Earth	$5.98 \times 10^{24}\ \mathrm{kg}$
mass of the Moon	$7.35 \times 10^{22}\ \mathrm{kg}$
average radius of Earth orbit	$1.50 \times 10^{11}\ \mathrm{m}$
average radius of Moon orbit	$3.84 \times 10^{8}\ \mathrm{m}$

Useful conversions

1 degree ≈ 0.0174 5 radian

1 radian ≈ 57.30 degrees

absolute zero: 0 K = −273.15 °C

1 electronvolt (eV) = 1.602×10^{-19} J

SI units and conversions

[The base units are: m; kg; s; A; K; mol; cd]

Quantity	Unit	Conversion
speed	$\mathrm{m\ s^{-1}}$	
acceleration	$\mathrm{m\ s^{-2}}$	
angular speed	$\mathrm{rad\ s^{-1}}$	
angular acceleration	$\mathrm{rad\ s^{-2}}$	
linear momentum	$\mathrm{kg\ m\ s^{-1}}$	
angular momentum	$\mathrm{kg\ m^2\ s^{-1}}$	
force	newton (N)	$1\ \mathrm{N} = 1\ \mathrm{kg\ m\ s^{-2}}$
energy	joule (J)	$1\ \mathrm{J} = 1\ \mathrm{N\ m} = 1\ \mathrm{kg\ m^2\ s^{-2}}$
torque	N m	
power	watt (W)	$1\ \mathrm{W} = 1\ \mathrm{J\ s^{-1}}$
pressure	pascal (Pa)	$1\ \mathrm{Pa} = 1\ \mathrm{N\ m^{-2}}$
frequency	hertz (Hz)	$1\ \mathrm{Hz} = 1\ \mathrm{s^{-1}}$
charge	coulomb (C)	$1\ \mathrm{C} = 1\ \mathrm{A\ s}$
potential difference	volt (V)	$1\ \mathrm{V} = 1\ \mathrm{J\ C^{-1}}$
resistance	ohm (Ω)	$1\ \Omega = 1\ \mathrm{V\ A^{-1}}$
capacitance	farad (F)	$1\ \mathrm{F} = 1\ \mathrm{A\ s\ V^{-1}}$
inductance	henry (H)	$1\ \mathrm{H} = 1\ \mathrm{V\ s\ A^{-1}}$
conductivity	siemens (S)	$1\ \mathrm{S} = 1\ \Omega^{-1}\ \mathrm{m^{-1}}$
electric field	$\mathrm{N\ C^{-1}}$	$1\ \mathrm{N\ C^{-1}} = 1\ \mathrm{V\ m^{-1}}$
magnetic field	tesla (T)	$1\ \mathrm{T} = 1\ \mathrm{N\ s\ m^{-1}\ C^{-1}} = 1\ \mathrm{kg\ s^{-2}\ A^{-1}}$
magnetic flux	weber (Wb)	$1\ \mathrm{Wb} = 1\ \mathrm{T\ m^2}$
angular frequency	hertz (Hz)	$1\ \mathrm{Hz} = 1\ \mathrm{s^{-1}}$
angular wavenumber	$\mathrm{m^{-1}}$	
activity	becquerel (Bq)	$1\ \mathrm{Bq} = 1\ \mathrm{s^{-1}}$
absorbed dose	gray (Gy)	$1\ \mathrm{Gy} = 1\ \mathrm{J\ kg^{-1}}$
dose equivalent	sievert (Sv)	$1\ \mathrm{Sv} = 1\ \mathrm{J\ kg^{-1}}$

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